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A STUDY OF THE POLLUTION AND NATURAL PURIFICATION OF THE OHIO RIVER. III. FACTORS CONCERNED IN THE PHENOMENA OF OXIDATION AND REAERATION

H. W. Streeter, et al

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A STUDY OF THE POLLUTION AND NATURAL PURIFICATION OF THE OHIO RIVER

III. FACTORS CONCERNED IN THE PHENOMENA OF OXIDATION AND REAERATION

By

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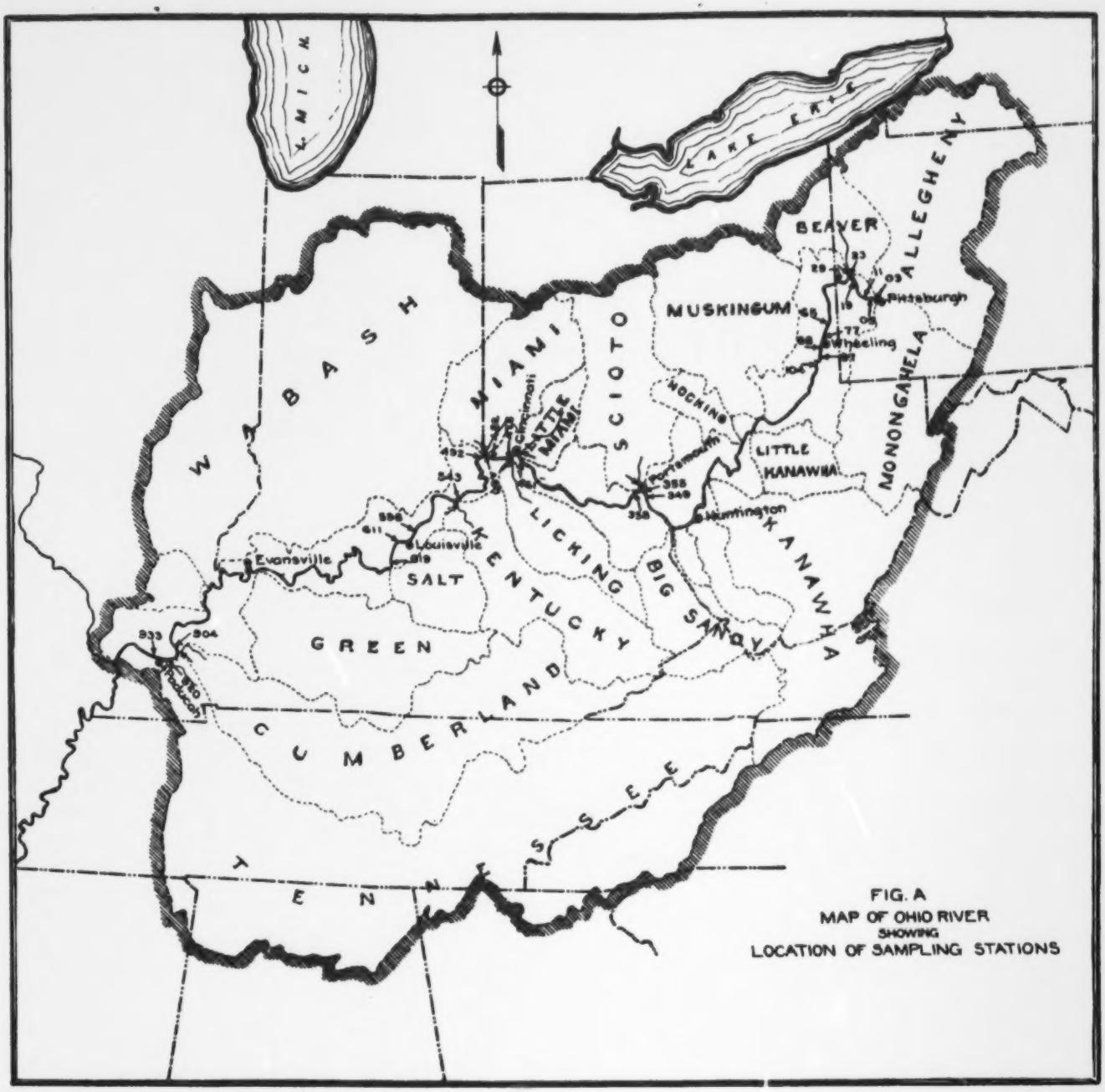

FIG. A
MAP OF OHIO RIVER
SHOWING
LOCATION OF SAMPLING STATIONS

FACTORS CONCERNED IN THE PHENOMENA OF OXIDATION AND REAERATION[1]

INTRODUCTION

The studies presented in this report were made by the United States Public Health Service during the years 1914 and 1915 as part of a comprehensive survey of the pollution and natural purification of the Ohio River, conducted under the supervision of Surg. W. H. Frost. The scope and purposes of this survey as a whole are outlined in a previous publication,[2] to which reference is made for a detailed description of the Ohio River, summaries of its sources of pollution, and measurements of discharge and velocity, also for presentation and discussion of the results of bacteriological examinations and chemical analyses other than those dealing with observations on dissolved oxygen. These latter observations, which are presented in the previous publication only in the form of a basic summary[3] without discussion, are regarded as being sufficiently distinctive in their significance and in the character of analysis required to justify their separate treatment in this supplementary report.

In the case of a stream like the Ohio River, important not only for its size and navigability but also because it is the sole available source of water supply for a large and growing population, permissible limits of its pollution are given in bacteriological rather than chemical terms; that is, they are fixed by the permissible load of bacterial pollution which may be placed upon the river with reference to water supplies rather than by considerations involving the exhaustion of the oxygen supply and physical nuisance. Nevertheless, the possibilities existing for at least a partial depletion of the reserve oxygen supply of the river, under conditions of pollution already approaching a critical stage from a bacterial standpoint in certain zones, could hardly be neglected in any consideration of the general problem. An exceptional opportunity was presented, moreover, for a study of the oxygen status of the river, and especially of its purification capacity from an oxygen standpoint, in view of the availability of extensive collateral data relative to river temperature, discharge,

[1] Manuscript submitted June, 1924.

[2] A Study of the Pollution and Natural Purification of the Ohio River. II. Report on Surveys and Laboratory Studies, Public Health Bulletin No. 143, Washington, 1924.

[3] Op. cit., Table No. 51, pp. 124–128.

and times of flow between the various sampling stations, collected in connection with the investigation.

The general viewpoint of the studies here described was influenced to a large extent by the newer conception of stream purification which has resulted from the marked advances that have taken place during recent years in the application of bacteriology and of physical and biological chemistry to sanitary science. First in importance have been discoveries relative to the generality of laws governing the death rates of bacteria, which have emphasized the progressive character of the complex biochemical reactions concerned in stream purification, and hence the controlling influence exerted by the time factor in such phenomena. Of almost equal significance has been the evolution of a newer biochemistry of sewage and sewage-polluted waters, wherein the older sewage chemistry, dealing with nitrogen in its various forms, has been largely replaced, in problems involving the stability of organic matters of sewage origin, by biochemical methods of study permitting a direct measurement of the oxidation reactions more directly related to organic stabilization processes. As examples of these methods may be cited the "relative stability" and "biochemical oxygen demand" tests, which are familiar to everyone who has followed the literature of sewage during recent years. Finally, the great value of modern physical chemistry as an aid in interpreting and applying to stream conditions the results of biochemical methods of study should be noted.

Previous to the foregoing developments, studies of the self-purification of streams had necessarily been, in the main, empirical; that is, they had comprised the determination and recording of actual conditions measured in analytical terms, without any attempt to formulate results in terms of general principles. Such a procedure is valuable as a matter of historical record relative to a given stream or local condition, but unfortunately it fails to give data having possibilities of more general application. In some cases generalizations such as have been attempted from data of this character have resulted in serious misconceptions of the relative importance of dilution as a factor in the oxidation of waste matters in streams, as compared with that of reaeration, which is often of far greater significance. A simple example will illustrate this point.

Accepted standards for a safe dilution ratio, based on certain stream conditions, especially in Massachusetts, are from a minimum of 3.5 to a maximum of 6.0 second-feet of normal unpolluted stream water per thousand of population contributing sewage.

The sewage of the District of Columbia has in summer a total biochemical oxygen demand equivalent to 112 grams per capita daily, or about 300 parts per million when corrected to a normal sewage strength basis of 100 gallons per capita flow, which is equivalent to

roughly 0.15 second-feet per thousand of population. These oxygen demand values, which agree very closely with the figure, 100 grams per capita daily, given by Pearse[4] as a result of studies of Chicago sewage, may be taken as being fairly representative for normal domestic sewage.

During the summer period, June 1 to October 15, 1914, the mean discharge of the Ohio River at a point immediately below Cincinnati represented a flow of 5.1 second-feet per thousand of urban population in the watershed above this point, at which point the average dissolved oxygen content observed was 5.7 parts per million, or 68 per cent saturation. Assuming the normal summer oxygen content of a theoretically unpolluted Ohio River to be at the mean summer saturation value, 8.2 parts per million, the draft imposed upon the normal dissolved oxygen content of the river to satisfy the oxygen demand of the urban sewage alone would be represented, in this case, by the difference between 8.2 and 5.7 parts, or 2.5 parts per million. On this basis, the required dilution, in the absence of reaeration, would be 300:2.5, or 120:1, amounting to a stream flow of 18.5 second-feet per thousand of population. Since the flow was actually but 5.1 second-feet per thousand, it would appear that dilution alone was responsible for about one-quarter, and reaeration for nearly three-quarters, of the total purification of the river taking place up to the point in question. With a permissible oxygen content of the Ohio River at this point lower than the approximately 70 per cent saturation figure observed during the summer of 1914, the stream flow actually required would be reduced by even more than a proportionate amount, as the relative quantities of dissolved oxygen supplied by reaeration would be greater with lower saturation values in the river.

The foregoing example not only illustrates the great importance of reaeration in the purification of running streams, but also emphasizes the fallacy of depending solely upon mere dilution as a measure of the oxidation of wastes discharged into flowing bodies of water. An attempt to apply dilution ratios, derived from rivers having great reaeration capacities, to sluggish streams like the Chicago drainage canal, would unquestionably lead to serious error, as has been borne out by experience with a number of sluggish canals, which have given rise to offensive conditions, though not excessively polluted from a standpoint of ordinary dilution criteria.

The development of a better understanding of the relations between dilution and reaeration as factors in stream purification, and in fact our modern conception of the overwhelming importance of dissolved oxygen in determining the power of natural bodies of water to digest and oxidize organic polluting matters, owes much to

[4] Trans. Am. Soc. C. E., vol. 85, 1922, p. 451.

the pioneer work in sewage bio-chemistry carried out by the Franklands, McGowan, Letts, Dibdin, Adeney and their coworkers in England, and to adaptations and modifications of their methods of study by various workers in the United States, notably Black and Phelps, Hoover, Lederer, and Theriault. While no extended bibliography of the studies above noted will be presented in connection with the present paper, a number of references covering certain specific points will be given in the later text.

With the aid of the newer bio-chemistry of sewage and of the quantitative view of the oxygen relations in polluted streams which it permits, it has been possible to formulate and test empirically a general theory of stream purification from an oxygen standpoint, using for this purpose the data obtained in connection with the Ohio River studies. Before proceeding with a presentation and analysis of these data, it will clarify the subsequent discussion to outline this underlying theory.

THEORETICAL DISCUSSION

The capacity of a stream to receive and oxidize sewage depends upon its oxygen resources. The condition of a polluted stream at any time is the result of a balance between these resources and the demand made upon them by the organic polluting matter carried by the stream. This demand, being the result of a slow bio-chemical reaction, is, in the absence of new pollution, a progressively decreasing one, and as the resources of the stream are composed in part of a continuous influx of oxygen from the atmosphere, the state of balance which determines the momentary condition of the stream is constantly changing. There are, therefore, two primary phases of the problem—namely, the actual, momentary condition, and the direction and extent of the existing changes, which indicate the future condition. Fresh sewage, for example, may contain some dissolved oxygen, and, measured upon the oxygen scale of nuisance, may be in the same momentary condition as a stream which has about completed the work of oxidizing organic pollution and contains the same amount of residual dissolved oxygen. The direction of change, however, is entirely different and determines the distinction between the two cases. The oxygen resources are comparable to the assets of a balance sheet; the oxygen demand to the liabilities. The condition of a strong sewage containing oxygen is comparable, in financial terms, to one of momentary solvency, with available cash, but with excessive obligations maturing on the morrow. A comprehensive theory of self-purification must therefore deal with the oxygen demand as well as with the oxygen resources, and must consider the relation of the various factors of time, temperature and other physi-

cal conditions to the rates of change of these two fundamental quantities.

THE OXYGEN DEMAND UPON A STREAM

Changes in the dissolved oxygen content of a stream are intimately associated with biochemical changes. They are brought about primarily by the oxidation of organic matter discharged into streams as soil wash and as wastes. In the presence of a supply of oxygen, together with certain oxidizing bacteria and oxidizable organic matter, progressive oxidation and stabilizing of the organic matter will take place.

It has been shown[1] that, under experimental conditions approximating those prevailing in a stream containing reserve dissolved oxygen, this reaction is an orderly and consistent one, proceeding at a measurable rate and according to the following definite law:[2]

The rate of biochemical oxidation of organic matter is proportional to the remaining concentration of unoxidized substance, measured in terms of oxidizability.

This law, which happens to be similar to that which defines the course of a monomolecular reaction,[3] may be stated in its differential form thus:

$$-\frac{dL}{dt} = KL$$

which may be integrated to the form—

$$\log \frac{L'}{L} = Kt$$

L′ being the initial and L the final oxidizability, or oxygen demand of the organic substance, in terms of oxygen; t being the elapsed time and K a constant coefficient, defining the rate at which the reaction proceeds. The value of K depends upon the character of organic matter and upon the temperature.

It will be assumed for the present that this relation holds also under actual stream conditions. Evidence supporting this assumption will be presented later in the text (p. 40).

Defining the oxygen demand as the total remaining oxidizability of the substance present at any time, the law states that in equal

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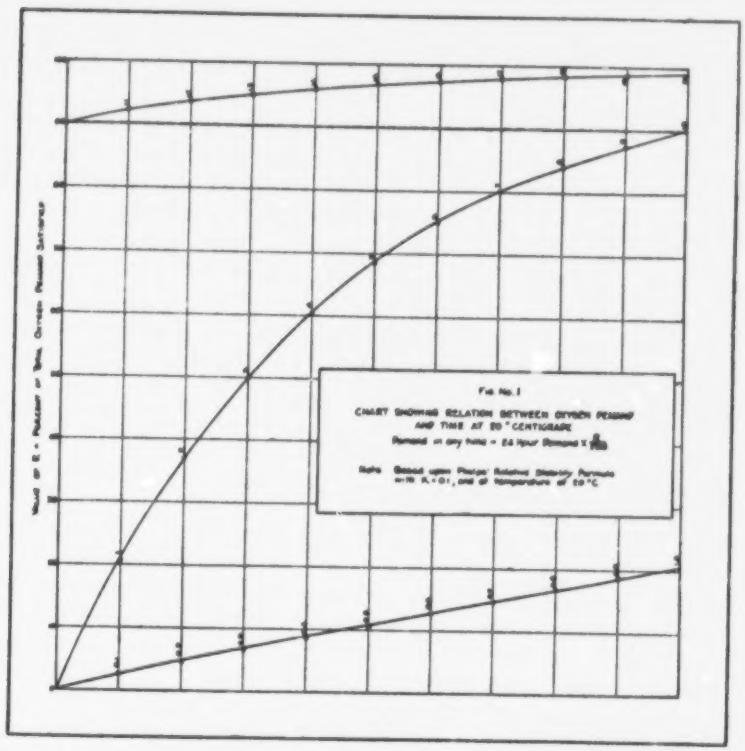
[1] Phelps, Earle B., Biochemistry of Sewage, VIII, Int. Cong. Appl. Chem., XXVI, 251.

[2] This statement should be qualified to the extent of noting that little definite knowledge exists as to whether the law stated holds for periods of time longer than about 20 days. Experimental data bearing on this point are, in fact, somewhat meager for periods longer than 10 days, though for shorter periods the most reliable evidence has been confirmatory.

[3] The similarity of this law to that of monomolecular chemical reactions is probably due to the biochemical nature of the phenomenon. If the reaction were strictly chemical it would theoretically be more likely to follow the law of bimolecular reactions, as the two reacting substances, oxygen and oxidizable organic matter, are both present in limited amounts.

periods of time an equal proportion of the remaining oxygen demand will be satisfied. That is, if 20 per cent of the initial oxygen demand be satisfied in the first 24 hours, 20 per cent of the remaining demand will be satisfied in the second 24 hours, and so on. (See fig. 1 for graphical illustration.) Since the oxygen demand, as actually determined in the laboratory, is given in terms of



dissolved oxygen, the rate of satisfaction of the demand, which is denoted by the term $\left(-\frac{dL}{dt}\right)$, is equivalent to the rate of oxygen depletion. If the dissolved oxygen content be expressed in terms of saturation deficit (D_1), its rate of depletion is $\left(\frac{dD_1}{dt}\right)$ and the following relation therefore holds:

$$-\frac{dL}{dt}=\frac{dD_1}{dt}=KL$$

The nature of the substances found in city sewage and the number and kinds of bacteria present have been found to be sufficiently

constant under various conditions to give a fairly constant value of K for this reaction at a constant temperature. Its value at 20° C., the time being expressed in days, was determined by Phelps by the methylene blue reaction for the sewage of Boston,[4] and later by direct determination of the oxygen demand of the sewage of Washington; and also computed from results obtained by Lederer[5] at Chicago in a comparison of the dilution and saltpeter incubation methods, using methylene blue. From all these data, values of K in the above equation of approximately 0.1 were obtained or derived. Later work by Lederer and collaborators[6] has indicated some variation of value of K for different sewages, though these differences are not striking. Values of the constant for different organic industrial wastes appear to vary more, doubtless because of the widely varying character of the organic matter present in such wastes, though the results of recent studies by Theriault[7] indicate that when tests are carefully controlled values of the constant similar to those for sewage are obtained for a wide class of industrial wastes. Values for polluted river water should not vary greatly from those for average sewage, since a major part of the oxidizable organic matter present in such water has its origin in sewage.[8]

For any particular sewage or similar waste the value of the velocity constant K is a function of the temperature. The temperature relation has been found empirically to be defined by the formula:

$$\frac{K'}{K} = \Theta^{(T'-T)}$$

in which (T′) and (T) are the two temperatures, (K′) and (K) the corresponding values of the velocity constant of the reaction, and Θ the thermal coefficient, a constant for the reaction, which may be determined experimentally.

The value of the thermal coefficient Θ has been determined experimentally at Cincinnati in connection with the present studies and also by Phelps in connection with studies at Boston and at the

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[4] Phelps, Earle B., U. S. Geol. Survey, W. S. Paper No. 229, Washington, 1909.

[5] Lederer, A., Jour. Inf. Dis., Vol. XIV, 1914, 482.

[6] Lederer, A., Am. Jour. Publ. Health, Vol. V, 1915, p. 354.

[7] Public Health Reports, Reprint No 594 (1920), U. S. Public Health Service, Washington, D. C.

[8] Later studies of the value of K, made on waters of the Chicago drainage canal, the upper Illinois River, and the Ohio River at Cincinnati, have given results closely confirming this statement. The studies on Ohio River water, which were in progress at the time of final revision of this text, have involved probably the most elaborate and carefully controlled series of tests thus far made of the form and constants of the oxidation curve, employing three separate temperatures of incubation, 8° C., 20° C., and 30° C. For periods of time up to about 20 days these curves have been found to follow closely the law stated on page 5, and their (K) values at 20° C. to agree within from 5 to 10 per cent with the value, 0.1, employed in connection with the present text.

Hygienic Laboratory at Washington. These values are given below, as follows:

	Value of Θ
1. Cincinnati (Mill Creek water):	
Temperature range, 10° C. to 20° C	1.0524
Temperature range, 20° C. to 37.5° C	1.0504
2. Boston (sewage, methylene blue method):	
Temperature range, 20° C. to 37.5° C	1.0415
3. Hygienic Laboratory (nitrate method):	
Temperature range, 15° C. to 24.5° C	1.0441
Temperature range, 24.5° C. to 37° C	*1.01
Average, excluding value marked (*)	1.047

For the purpose of the present discussion we will adopt the mean value of 1.047, obtained from the data given in this table, after eliminating the value of 1.01, which is obviously out of harmony with the others. The variation in the value of (K) at different temperatures, as defined by a value of Θ equal to 1.047 is illustrated in Figure No. 2, by the line showing the relation of K values at various temperatures to its value at 20° C. Conversion of the K values from one temperature to another may be readily made by means of a similar chart.

As stated on page 7 (footnote 8), further studies of the temperature relations of (K) for Ohio River water at Cincinnati, Ohio, were in progress at the Cincinnati laboratory of the Public Health Service at the time of final revision of the present text. From the results of these studies available up to that time, it appeared that values of the thermal correction factor Θ would be obtained agreeing closely with the mean value, 1.047, as above derived, assuming the experimental values of (K) at the different incubation temperatures to be computed in the same manner as in the present case. The method of computation thus followed has consisted of calculating, for the oxidation curve obtained at each respective incubation temperature, the percentage of the 20-day oxygen demand at that temperature remaining at successive intervals of time up to 20 days. These values, when plotted as ordinates against incubation times, using semilogarithmic paper, define a straight line having a slope equal to (K). This procedure carries an assumption that, for all practical purposes, the oxygen demand satisfied in 20 days is equivalent to the total available oxygen demand (with a value of K at 20° C. equal to 0.1, it is actually 99 per cent of the total). This assumption is substantially correct for temperatures of 20° C. and upwards, but becomes less so as the temperature is diminished below 20° C., since at the lower temperatures the time required for the satisfaction of a large proportion of the total oxygen demand is longer than at 20 degrees. When the studies described in the present text were made, no basis existed for extrapolating oxidation curves

beyond a time of 20 days at any temperature; hence, the coefficient (K), whether uncorrected at 20° C. or corrected to its equivalents at other temperatures, can hardly be regarded as being well defined for any period of time exceeding 20 days. This limiting time, however, usually is long enough for ordinary purposes of calculation.

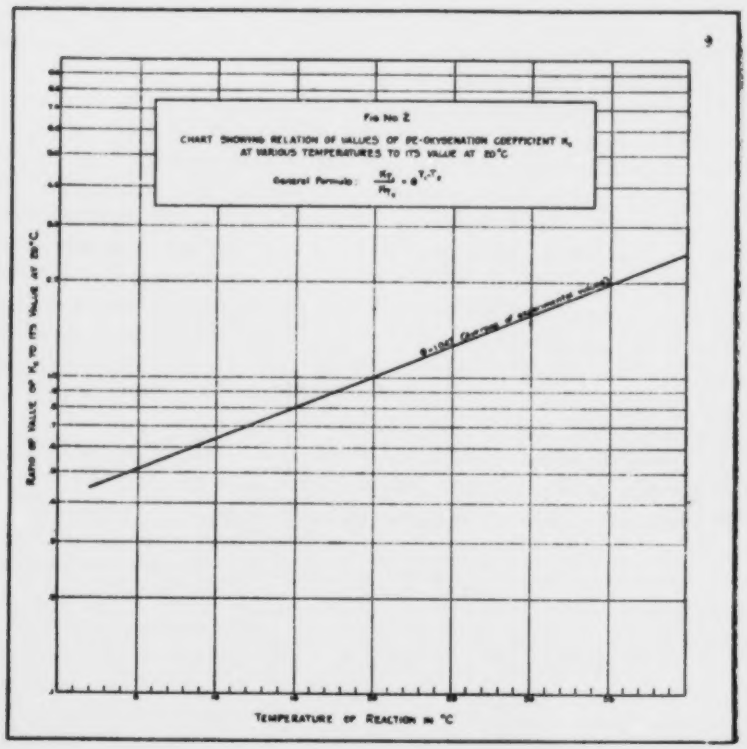
Fig No 2
CHART SHOWING RELATION OF VALUES OF RE-OXYGENATION COEFFICIENT K_2 AT VARIOUS TEMPERATURES TO ITS VALUE AT 20°C
General Formula: $\frac{K_2}{K_{20}} = \theta^{T-20}$
Ratio of Value of K_2 to its Value at 20°C
Temperature of Reaction in °C

THE OXYGEN RESOURCES OF A STREAM

Unpolluted water tends to hold in solution the maximum amount of oxygen which it is capable of containing at the existing temperature and partial oxygen pressure of the atmosphere. This is the so-called saturation value[9] and ranges at normal sea level barometric pressure from approximately 14 mg. per liter at just above freezing to about 7.6 mg. per liter at 30° C.

In polluted streams the draft imposed upon the dissolved oxygen supply by the progressive satisfaction of the oxygen demand reduces

[9] The actual saturation value depends upon the partial solution pressure of oxygen in the overlying atmosphere, and is directly proportional to the percentage of oxygen in the overlying gas. For normal atmospheric air, the solution pressure, and therefore the oxygen saturation value, is approximately one-fifth its value with an overlying atmosphere of pure oxygen.

the oxygen content below its saturation value; but as soon as this depression occurs absorption of oxygen from the atmosphere follows. The sources of available oxygen in a stream are therefore mainly two—the oxygen initially dissolved in the water, with its maximum possible equilibrium value at the saturation point, and the oxygen absorbed from the atmosphere by partially deoxygenated water in the process of physical reaeration.[10] Reaeration is a factor of so great importance in the biochemical oxidation of a stream that it merits discussion in some detail.

In a quiescent, unsaturated body of water the surface film, in immediate contact with the atmosphere, rapidly becomes saturated with oxygen, and can dissolve no more of the gas until that which has already passed into solution at the surface has been carried to the deeper strata by diffusion or "streaming,"[11] or by both of these agencies combined. A more or less well defined concentration gradient is thus formed, with approximate saturation at the surface and decreasing concentrations at increasing depths, the draft on the surface film being balanced by replenishment of the supply through solution from the atmosphere. Thus the rate at which a given small volume of water becomes reoxygenated is a function of its saturation deficit and its distance from the surface.

In a flowing stream conditions of quiescence are, of course, modified greatly by the constant vertical mixing or overturning which the water receives as it flows along the channel. An approach to quiescence may be found in occasional pools during periods of sluggish flow, but even under these circumstances slight currents and eddies, due to movements of the water and to slight differences in temperature, bring about a more or less continuous mixing of the stream throughout its cross section. The concentration gradient characteristic of quiescent waters is therefore subject to constant disturbance, but since a comparatively small proportion of the water reaches the exact surface layer, the forces distributing the dissolved oxygen throughout the body of the stream are those of mechanical mixing and diffusion combined. Viewed as a diffusion phenomenon, this distributing action may be conceived as being one in which the effect of mixing is simply to greatly decrease the effective depth through which diffusion acts, the amount of contraction in the effective depth being a function of the turbulence of the stream and de-

[10] A third possible source of available oxygen is the activity of certain aquatic plants, notably the green and blue-green algae, which occasionally find in sluggish pools a favorable environment for their propagation. This, however, is a special case which is not ordinarily an important factor in the reaeration of flowing streams.

[11] The term "streaming" has been applied by Adeney to the downward movement of currents of water from the surface to the lower strata, due largely, as subsequently noted by Phelps (Trans. Am. Soc. C. E., vol. 76, p. 1626), to slight differences in density resulting from surface evaporation.

pending upon the various physical characteristics of channel and flow effecting turbulence.

The conception that diffusion is the active agent in the distribution of dissolved oxygen throughout a turbulent stream, in the presence of what might appear to be the overwhelmingly superior forces of mechanical mixing, is a rather novel one and perhaps not readily obvious. Mechanical mixing of the water constantly breaks up the saturated surface film into small masses of water which are brought into contact with less saturated masses with an intimacy and frequency depending upon the degree of agitation. But the oxygen cannot pass from one zone to another, no matter how thin the zone or intimate the mixture of particles, except by diffusion. If this force were absent, the effect of mixing, whether mechanically or by "streaming," would be similar to that of combining a measure of peas and beans, the aggregation being merely one of saturated and unsaturated particles of water. The laws of diffusion therefore have an important bearing upon the phenomenon of reaeration, even in running streams.

The process of diffusion had been studied by many of the earlier chemists, but the first attempt to formulate a definite principle was made by Fick,[12] who enunciated what has since been known as Fick's law of hydrodiffusion, as follows:

> A dissolved substance tends to diffuse through the solvent from a region of greater to one of less concentration, and the amount of substance which passes through a unit cross section in unit time is proportional to the difference in concentration on either side of this section.

This law in its general form is capable of expression only as a differential equation, which requires for its solution the establishment of certain restricting or boundary conditions.

A mathematical analysis of the law of Fick was made by Black and Phelps[13] for the following special case. A column of water of stated depth initially contains dissolved oxygen at a stated concentration, less than that of saturation, and uniformly distributed throughout the length of the column. At the initial time this column is exposed at its surface to the atmosphere the process of downward diffusion is inaugurated and a concentration gradient established from surface to bottom. Required, the average concentration of oxygen throughout the column at the end of a stated time.

The final integrated equation which was obtained is—

$$D = 100 - \left(1 - \frac{B}{100}\right)\left(81.06\right)\left(e^{-K} + \frac{e^{-9K}}{9} + \frac{e^{-25K}}{25} + \frac{e^{-49K}}{49} - - -\right)$$

[12] Pogg. Ann., 94, 1855, p. 59. Nernst, Theoretische Chemie, 2d German ed., p. 157.
[13] Black, Col. W. M., and Phelps, E. B., The Discharge of Sewage into New York Harbor. Report made to the Board of Estimate and Apportionment, New York City, 1911.

in which—

D=the required average concentration of dissolved oxygen expressed in per cent of saturation, after time, t.

B=the uniformly distributed initial concentration expressed in the same units.

e=2.718.

$K=\frac{a\ t\pi^2}{4\ L^2}$

t=the elapsed time since concentration (B) existed.

L=the linear depth of the column.

a=the diffusion coefficient, a constant for pure water, and a given gas (oxygen) at constant temperature.

This equation applies primarily to quiescent water, but it has already been indicated that the effect of mixing is virtually to decrease the effective depth through which diffusion acts, so that the application to turbulent waters is permissible, the depth term being then indeterminate.

For the purposes of the present discussion it is desired to determine the rate of solution of oxygen under any stated conditions. This rate is derivable from the equation of condition by differentiation with respect to a variable time. The actual operation need not be performed, for it is obvious that the result will be a complex expression containing the factor $\left(1-\frac{B}{100}\right)$, the initial saturation deficit, together with a complex time function. The rate of solution, therefore, is proportional, among other things, to the initial saturation deficit, and the proportionality factor is itself a time function—i. e., it varies with the time. At zero time—i. e., at the start— the rate is strictly proportional to the deficit.

In any stream having sufficient turbulence to make reaeration a real factor in self-purification, there is sufficient mixing to bring about a fairly uniform condition throughout the vertical cross-section, so that at no time is there material vertical stratification. Such a condition may be regarded as being made up of short periods of quiescence followed by complete mixing, so that the final dissolved oxygen value at the end of any short period of quiescence becomes the initial, uniformly distributed value at the beginning of the next period. Since the rate of solution at the beginning of any period is strictly proportional to the deficit at that time, it follows that under the assumed conditions—and these are approximately the conditions met with—the rate of solution is proportional to the existing saturation deficit. This is the law of the velocity of solution of solids in water, derived experimentally by Noyes and Whitney.[14]

[14] Noyes, A. A., and Whitney, W. R., Zeit. f. physik. Chem., 23, 1897, p. 689.

The physics of reaeration phenomena have been studied extensively by Adeney [15] and his coworkers, who have developed the following formula expressing the rate of reaeration of quiescent columns of water:

$$w = (100 - w_1)\ (1 - e^{-f(a/v)t})$$

where—

w=amount of gas dissolved, expressed as percentage of saturation.
w_1=initial concentration.
f=coefficient of escape of gas from the liquid per unit of area and volume.
v=volume of liquid.
a=area of surface.
t=time of exposure.

If this formula be differentiated with respect to time (t), it becomes—

$$\frac{dw}{dt} = (100 - w_1) \cdot f(a/v) \cdot e^{-f(a/v)t}$$

whence the rate of reaeration $\left(\frac{dw}{dt}\right)$ is shown to be directly proportional to the saturation deficit $(100-w_1)$. It is thus apparent that whether reaeration be viewed as a phenomenon of diffusion or one of "streaming," both theories lead to the same conclusion as regards the fundamental importance of the law of solution in determining the rate of progression of the reaction.

An experimental confirmation of this law as applied to the solution of atmospheric oxygen by water was obtained by Dibdin[16] from a series of about 150 tests made in connection with experiments on the condition of the water in the River Thames, in England. He exposed deaerated water in open vessels for periods of time ranging from 1 to 96 hours, determining the dissolved oxygen at intermediate intervals. The results obtained were plotted in the form of a curve, which has been reproduced in a different form in Figure No. 3, the ordinates being saturation deficit values plotted on a logarithmic scale and the abscissae, corresponding times from the starting point. It is noted that the plotted observations lie almost exactly along a straight line. Denoting the time as (t) and the saturation deficit as (D) we have, then, that—

$$\log D = -(c\ t + d)$$

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[15] Adeney, W. E. Sci. Proc. Royal Dublin Soc., 1914.
Adeney, W. E., and Becker, H. G. Philosophical Magazine, vol. 38, 1919, pp. 317–338.
Philosophical Magazine, vol. 42, 1921, pp. 87–96.
Adeney, W. E., Leonard, A. S. G., and Richardson, A. Sci. Proc. Dub. Roy. Soc., vol. 17, 1922, pp. 19–28.

[16] Dibdin, W. J., The Purification of Sewage and Water, 3d. ed., 1903, pp. 283–284 (Diagram 8).

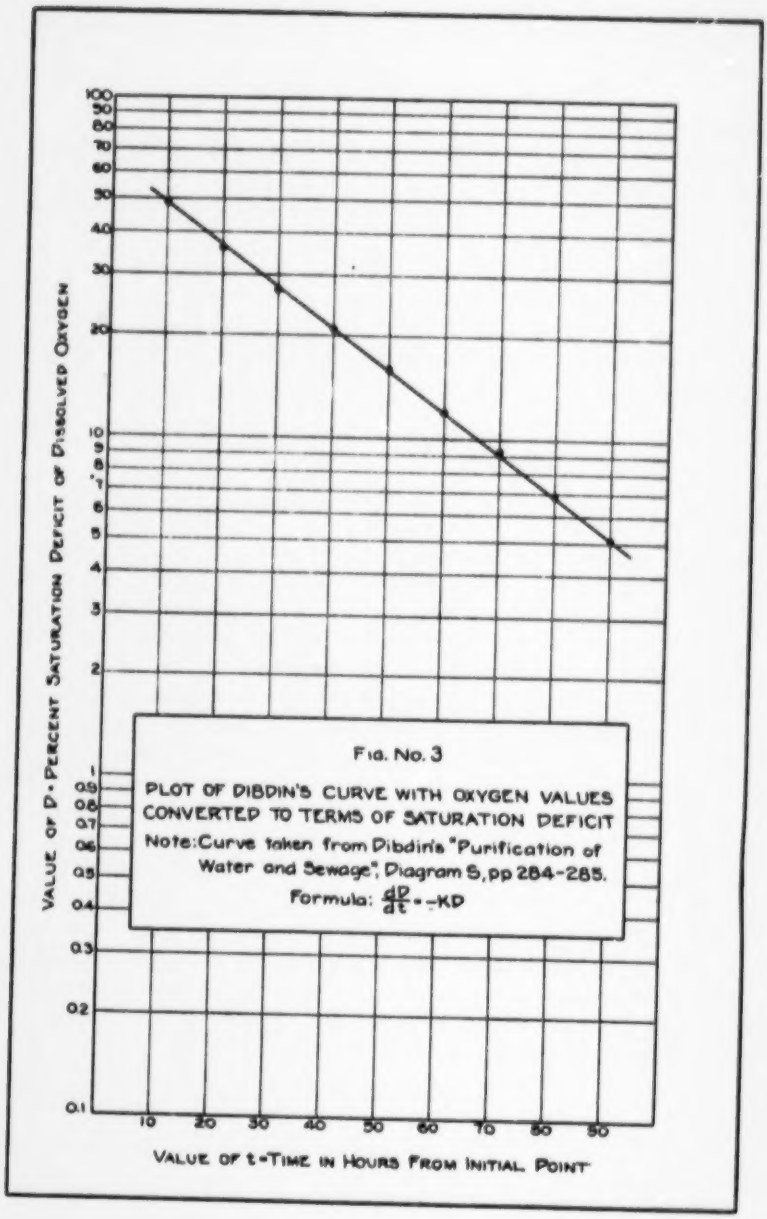


Fig. No. 3

PLOT OF DIBDIN'S CURVE WITH OXYGEN VALUES CONVERTED TO TERMS OF SATURATION DEFICIT

Note: Curve taken from Dibdin's "Purification of Water and Sewage", Diagram 5, pp 284-285.

Formula: $\frac{dp}{dt} = -KD$

the constant (c) being the slope of the line and (d) being the intercept on the (D) axis. Differentiating this equation with respect to (t), we have—

$$-\frac{dD}{dt}=cD$$

Since $\left(\frac{dD}{dt}\right)$ defines the rate of solution of oxygen in terms of saturation deficit, it is apparent that this rate is directly proportional to the saturation deficit (D), which is in accordance with the general law of solution above stated.

From what has been previously stated regarding the relation of mechanical mixing to diffusion and "streaming" as agencies in the reaeration of flowing bodies of water, it naturally follows that reaeration of streams, while conditioned primarily by the rate of solution of oxygen at the water surface, is modified by those factors which affect the rapidity and thoroughness with which the oxygen, once dissolved, is distributed throughout its depth. Thus it was found by Fair[17] that at a given saturation deficit value, the rate of reaeration of a quiescent body of water is greatly accelerated merely by the mixing resulting from induced wave action. In Sheffield, England, an activated sludge plant for sewage treatment has recently been developed in which the high rate of aeration required for oxidation of the sewage is obtained entirely by a mechanical mixing device. Examples of this kind, which are numerous, show conclusively the great influence exerted upon the rate of reaeration of flowing bodies of water by those forces of mixing and convection which may be summed up under the term "turbulence." In running streams the turbulence factor is highly variable and produces correspondingly varied effects upon reaeration rates. In a given stream stretch and under a given condition of flow, where the turbulence remains fairly constant, the rate of reaeration is a direct function of, and should be closely proportional to, the prevailing oxygen saturation deficit. Under conditions found in streams, therefore, the operation of the law of solution is fundamental, and variations in the rate of solution are governed largely by those physical characteristics of a given stream which cause different degrees of turbulence.

THE OXYGEN BALANCE IN A STREAM

The two opposing reactions, deoxygenation and reaeration, tend always to come to a condition of temporary equilibrium. If the water be nearly oxygen-saturated, and highly polluted, there is a

[17] Discussion of paper by R. H. Gould, "The area of water surface as a controlling factor in the condition of polluted harbor waters." Trans. Am. Soc. C. E., vol. 85, 1922, pp. 728–731.

rapid rate of withdrawal of oxygen and a slow rate of replacement, resulting in a decrease in the available dissolved oxygen. As this value decreases, the rate of reoxygenation is correspondingly increased until it equals the rate of depletion, at which point the two reactions are for the moment in equilibrium and there is no change in the actual oxygen content. This equilibrium, however, is ony momentary, for the decreasing oxygen demand of the organic matter, resulting from its own oxidation, makes the rate of depletion correspondingly less and permits the gradual recovery of the dissolved oxygen up to its full saturation value. Under conditions of continuous or repeated pollution, however, an equilibrium point may be reached at which the rate of reoxygenation is exactly equal to that of deoxygenation, and is so maintained. The importance of the reoxygenation factor itself and of its accurate experimental determination is therefore obvious. Upon this value, under any given conditions, depends the resultant oxygen condition of the stream for a stated degree of pollution, or, conversely, the maximum amount of pollution compatible with any stated degree of oxygen depletion.

From the primary laws of oxidation of organic matter and of reoxygenation of a stream, the resultant general equation of stream condition may now be derived, expressed in terms of dissolved oxygen. The application of the experimental data to this equation will then permit the derivation of the various constants which characterize the stream in regard to its capacity to receive and dispose of sewage pollution.

According to the argument which has been presented, the rate of change in the oxygen deficit is governed by two independent reactions. First, the deficit increases at a rate which may be assumed[18] to be proportional to the oxygen demand of the organic matter. Secondly, it decreases by reaeration, at a rate directly proportional to its own value. The two rates may be expressed in differential form thus:

$$-\frac{dL}{dt} = \frac{dD_1}{dt} = K_1 L \text{ (see p. 5)}$$

and

$$\frac{dD_2}{dt} = -K_2 D$$

in which—

t = time of reaction, in days.

L = oxygen demand of the organic matter, expressed in terms of parts per million of oxygen.

D = oxygen saturation deficit of the water, in parts per million.

[18] For evidence supporting this assumption see later text, pp. 40 to 44.

$\frac{dD_1}{dt}$ = rate of deoxygenation, in terms of oxygen saturation deficit.

$\frac{dD_2}{dt}$ = rate of reaeration, in terms of oxygen saturation deficit.

K_1 = coefficient defining the rate of deoxygenation.

K_2 = coefficient defining the rate of reaeration.

The net rate of change in the oxygen deficit (D) at any time is equal to the difference between (or algebraic sum of) the two par-

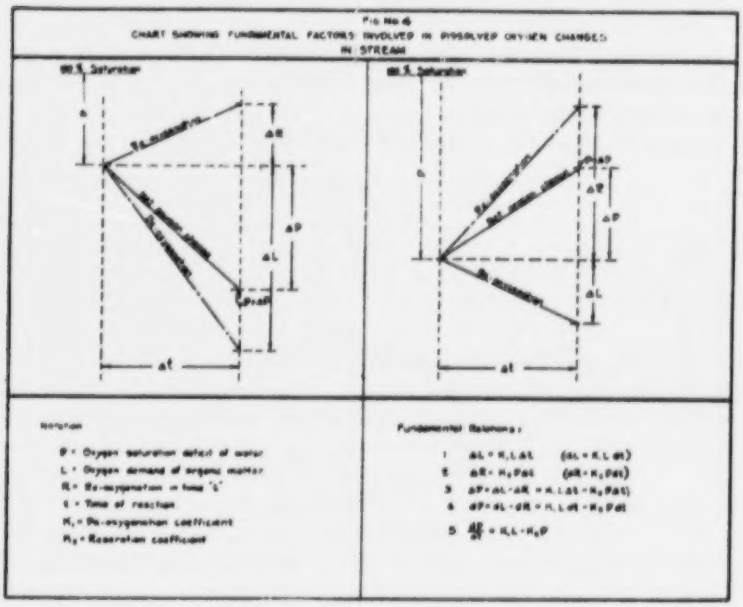


Fig No 4
CHART SHOWING FUNDAMENTAL FACTORS INVOLVED IN DISSOLVED OXYGEN CHANGES IN STREAM
% S. Saturation
% S. Saturation
Notation
D = Oxygen saturation deficit of water
L = Oxygen demand of organic matter
R = Re-oxygenation in time "t"
t = Time of reaction
K1 = De-oxygenation coefficient
K2 = Reaeration coefficient
Fundamental Relations:
1 ΔL = K1 L Δt (dL = K1 L dt)
2 ΔR = K2 Pdt (dR = K2 Pdt)
3 ΔP = ΔL - ΔR = K1 L Δt - K2 Pdt)
4 dP = dL - dR = K1 L dt - K2 P dt
5 dP/dt = K1 L - K2 P

tial and opposing rates as defined above (see fig. No. 4) and may be expressed mathematically as follows:

$$\frac{dD}{dt} = \frac{dD_1}{dt} + \frac{dD_2}{dt}$$

whence—

$$\frac{dD}{dt} = K_1L - K_2D$$

which is a linear differential equation of the first order (sometimes called Leibnitz's equation) having the general form—

$$\frac{dy}{dx} + Py = Q$$

The integrated equation derived from this differential equation defines the actual dissolved oxygen content of the water, expressed as the deficit below saturation, at any time, in terms of the other

factors. The full mathematical derivation of the integrated equation is given in Appendix A, the resulting formula being—

$$D = \frac{K_1 L_a}{K_2 - K_1}(e^{-K_1 t} - e^{-K_2 t}) + D_a e^{-K_2 t} \qquad (1)$$

in which—

D_a=initial dissolved oxygen saturation deficit of the water, in parts per million.

D=saturation deficit, in parts per million, after time (t).

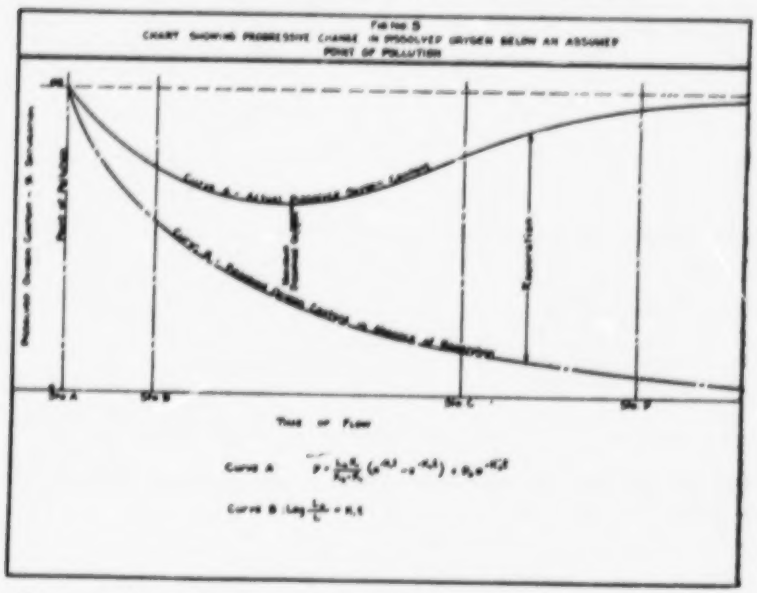
L_a=initial oxygen demand of the organic matter of the water, in parts per million.

K_1=coefficient defining the rate of deoxygenation.

K_2=coefficient defining the rate of reaeration.

t=elapsed time, in days.

e=base of Naperian or natural logarithms=2.71828.



Common logarithms may be used in obtaining values of (K_1) and (K_2), in which case the quantity (10) is substituted for (e). The quantities (D_a), (D), (L_a), and (L) may be expressed in terms of either parts per million or per cent of oxygen saturation; but should be stated invariably in the same terms.

The type of curve defined by this formula is illustrated by curve A in Figure No. 5, which is based on an assumed simple case wherein the water is saturated with oxygen initially and all of the polluting matter enters a given stretch of the stream at or above its upper limit (Station A). In the curve shown, however, the ordinates are

in terms of the oxygen saturation value, which is equal to 100 per cent minus the deficit (D) as given by formula (1). The curve, which is typical of oxygen conditions frequently observed in streams below major points of pollution, has a distinct minimum point, where the rates of deoxygenation and of reaeration are momentarily equal. By differentiating equation (1) with respect to time, and placing the resulting expression equal to zero, the point of minimum oxygen content, in terms of time (t), is thus defined:

$$e^{(K_2-K_1)t} = \frac{K_2}{K_1}\left[1 - \frac{D_a(K_2-K_1)}{L_aK_1}\right] \qquad (2)$$

It will be noted that in equation (2) the variable oxygen deficit, represented by (D) in equation (1), is absent. In order to obtain its value, equations (1) and (2) may be combined, giving the following expression:

$$\text{Log } D + K_2t = \text{Log}\left[L_a - \frac{D_a(K_2-K_1}{K_1}\right] \qquad (3)$$

This equation can be cleared of all terms in (t) and the maximum deficit expressed in terms only of the four constants, but the expression obtained is unwieldy, and it is more convenient to solve first for (t) in equation (2) and then for (D) in equation (3).

Similarly, the temperature function of this point may be obtained in a single equation, giving the time and maximum deficit as affected by variation in temperature, but this also is inconvenient for practical use, and the result is obtained more directly by using (K_1) and (K_2) values independently determined for the temperature in question. The temperature effect upon the constant (K_1) has already been given; that for the reaeration coefficient (K_2), which remains to be determined experimentally, will be discussed later in connection with the application of the experimental data to the determination of that constant.

The significance of the various terms in equation (1) is fairly obvious, excepting that of the reaeration coefficient (K_2), the meaning of the deoxygenation constant (K_1) having been previously discussed (p. 6). The reaeration coefficient (K_2) is analogous to (K_1) in that it defines a geometric rate of progression on a time basis; for example, if the value of (K_2) be such that 20 per cent of the existing saturation deficit is satisfied by reaeration in the first unit of time, then 20 per cent of the remaining deficit will be satisfied in the second unit of time, and so on. It differs from (K_1), however, in not being a constant for a given temperature, as is true (or approximately so) of (K_1). It has already been noted that the rate of reaeration of a body of water is modified to a large extent by its degree of turbulence, other things being equal. In flowing

streams, where turbulence undergoes wide variations according to velocity of flow, character, and slope of the channel, and other physical factors, correspondingly marked differences are to be expected in rates of reaeration and hence in values of (K_2), observed in different stretches of the same river, or even in the same stream stretch under varying flow conditions. On the other hand, the reaeration rate should be closely related to and governed by those conditions which influence turbulence of flow; hence values of (K_2), as actually determined in a given river stretch, should bear a close relation to the several measurable factors of physical condition which cause varying degrees of turbulence. With a given type of channel or flow condition, values of the coefficient (K_2) should be well defined and characteristic for that type or condition.

The practical significance of the reaeration coefficient in problems of stream pollution is therefore twofold. First, there is the local application, in which a set of (K_2) values, once determined for a particular stream, may be utilized to calculate its capacity for reaeration under any assumed conditions of future pollution, using for this purpose the three formulas that have been developed above. (See pp. 18–19.) Secondly, there is the more general application, wherein correlations of values of (K_2) with certain measurable factors of physical stream condition may be employed to estimate the reaeration capacities of other streams for which these factors are known but in which no direct measurements have been made of the coefficient. (See p. 64.) Obviously the point of departure for a study of this kind is the direct measurement of the value of (K_2) in the stream. The methods which have been employed in the derivation of reaeration coefficients for a number of stretches of the Ohio River will therefore be described in some detail.

Referring now to equation (1), if the problem were that of solving for an unknown oxygen content (D), with all of the other factors known, the matter would be one of simple substitution. This would be the procedure in practice, the constants being given and it being required to determine the residual dissolved oxygen after any time and under given or assumed conditions of pollution. In experimentally measuring the reaeration coefficient (K_2), however, all of the other terms, including (D), must first be determined. For convenience in reference, these terms will be listed again, as follows:

K_1=the deoxygenation constant.

L_a=the initial total oxygen demand of the stream water in parts per million of oxygen.

D_a=the initial dissolved oxygen content of the stream water in parts per million, expressed as saturation deficit.

D=the dissolved oxygen content of the stream, in parts per million, after a time "t" from the initial point.

t=the time of flow, in days, from an oxygen content of (D_a) to one of (D).

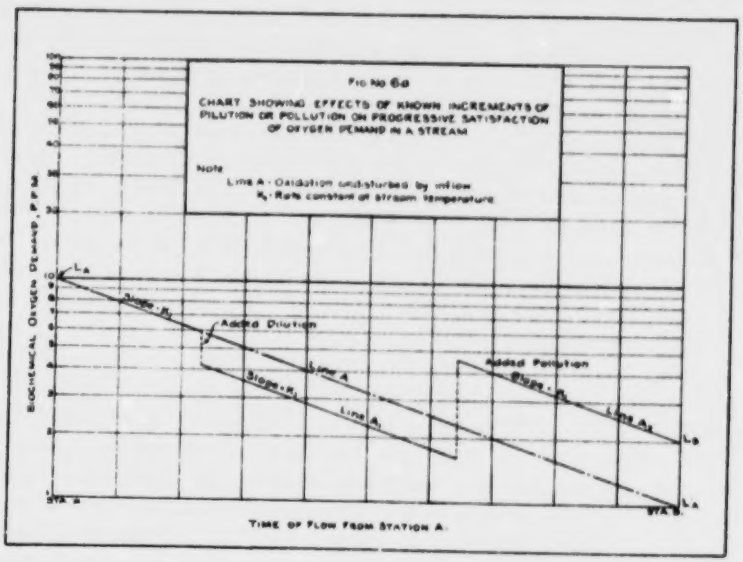
The determination of (K_1) and the temperature relations involved already have been discussed. In the present case values have been assigned to (K_1) by correcting its value, 0.100, at 20° C. to its various equivalents at different observed stream temperatures, using for this purpose the curve shown in Figure No. 2 or the correction formula given on page 7, with the thermal coefficient (θ) taken as equal to 1.047. The determinations of (D_a) and (D), and of the time (t) in a given stream stretch, require the selection of two sampling stations, A and B, one located at the upstream and the other at the downstream end of the stretch, and direct observation of the dissolved oxygen content and the water temperature at these two stations, together with measurements or computations of the mean time of flow between them at the various river stages at which the dissolved oxygen content is observed.

The assignment of a proper value to the initial oxygen demand (L_a), however, presents an extremely difficult problem, owing largely to the uncertainties involved in determining, under conditions found in natural streams, a quantity which may be taken as being a representative one for a particular river stretch under observation. It is proposed, therefore, to discuss this question in considerable detail.

If the river stretch in question were entirely free from inflowing pollution or dilution at points intermediate between two given sampling stations, or if the amounts and points of entry of such inflow were definitely known, the various factors concerned in the problem would be determinate and its solution would be comparatively simple. In the first case, the only change in the initially observed oxygen demand, occurring between the upper Station A and the lower Station B, would be a progressive and orderly decrease due to oxidation, following a course which, for all practical purposes, could safely be assumed to be similar to the logarithmic time function curve defined by line A in Figure No. 6a. The initially observed oxygen demand, denoted as (L_A) in the chart, therefore would become the value of (L_a) in formula (1). In the second case, where known increments of pollution or dilution entered the river stretch at recognized points between Stations A and B, the position of the line of residual oxygen demand would be altered at each one of these points, as shown in Figure No. 6a, but its slope, defining the rate of oxidation, would remain constant for the particular river temperature condition prevailing. In this latter instance, a new value of (L_a) would be computed at each point of entry of known inflow, the new ordinate of the curve lying above or below line A according to whether the effect of the added increment of

overflow at that point were one of added pollution or added dilution, respectively.

The condition actually encountered in a large majority of cases, however, is far from being the simple one above described, owing to the fact that the density of oxidizable matter carried by natural streams is subject to frequent disturbances due to the effects of inflow, sedimentation, and channel scouring. The combined influence of these disturbing factors may bring about marked changes in the oxygen demand of a stream wholly unrelated to those which may be caused by the orderly processes of oxidation; thereby masking the latter, in so far as they may be observable, and rendering uncertain



any attempted determination of a representative oxygen demand figure for a particular river stretch from observations at fixed sampling stations. In the absence of specific information as to the amounts and points of application of the various disturbing factors noted, the only clue as to their combined influence is afforded by the extent to which the oxygen demand (L_B). observed at station B (fig. No. 6a), deviates from the residual component (L'_A) of the initially observed oxygen demand (L_A) remaining at station B, assuming (L_A) to have been satisfied along line A in accordance with the law of oxidation stated on page 5, and making due allowance for any deviation due to errors of observation or extrapola-

tion.[19] The extent of divergence between (L_B) and (L'_A) gives no indication, however, as to the total amount of intermediate disturbance or the manner in which it may be distributed between A and B.

Under these circumstances, any value assigned to (L_a) for substitution in formula (1) must be an hypothetical one, based on some method of combining the results of observations at stations A and B in such a way as to give, as nearly as practicable, a measure of the total amount of oxygen demand satisfied between the two stations. With the foregoing limitations, it is fairly evident, however, that no single value of (L_a), assigned according to a standardized procedure, can exactly satisfy this requirement. Any procedure adopted, therefore, must be a compromise at best, designed to secure a fair approximation to the result sought.

There are three possible methods by which a value of (L_a) may be derived from observations of oxygen demand at A and B. Thus, (L_a) may be based on:

1. A mean[20] of the two observations (L_A) and (L_B).
2. The observed value either of (L_A) or of (L_B).
3. A mean[20] of the observed (L_B) and the residual component (L'_A) of the observed (L_A) remaining at station B.

Method (1) obviously is illogical in that it assumes a straight-line change in oxygen demand from a value of (L_A) at station A to one of (L_B) at station B and fails to take any account of the expected progressive decrease in (L_A) between the two points, due to oxidation. The sources of error which method (2) entails may best be appreciated by referring to Figure No. 6b, in which line A and line B illustrate the assumptions involved in adopting, respectively, an observed value of (L_A) or one of (L_B) as the hypothetical value of (L_a). If (L_A) is used for the purpose, any divergence between (L_B) and (L'_A) at station B is considered negligible, and a condition of zero disturbance is thus assumed. As previously noted, such a condition rarely, if ever, exists in natural streams. If (L_B) is employed, the amount of divergence between (L_B) and (L'_A) is assumed to be due to an initial oxygen demand equivalent to the quan-

[19] Errors of observation, or those involved in extrapolating the observed quantity (L_A) to its residual value (L'_A), at Station B, may cause a divergence of (L_B) from (L'_A) entirely dissociated from any deviation due to intermediate disturbance. The effect of observational errors should, in general, be so small as to be inappreciable. The effect of extrapolation errors, by which is meant principally those due to errors in the constants employed in calculating the value of (L'_A), might be sufficiently great, where the time factor were long, to cause an appreciable difference between this quantity, as calculated, and the observed value (L_B), owing to the fact that such errors become enlarged, in comparison with (L_B), when extrapolated over long periods of time. Any slight error in the assumed value of (K_1), for example, or in its temperature correction factor (Θ), might produce an effect of this kind if magnified over a long time period. For this reason, observations in river stretches of considerable length, where the period of flow exceeds several days, should be avoided where possible and their results interpreted with due regard for this source of error.

[20] The term "mean" is used here as signifying a simple arithmetical average. A weighted mean would hardly be justified in either case, owing to the absence of any definite basis for establishing relative weights.

tity (L'_B) in the chart, representing the maximum amount of disturbance which could possibly account for the divergence observed in a given case. The use of either (L_A) or (L_B) as a basis for (L_a) represents, therefore, the two most extreme assumptions possible regarding the amount of intermediate disturbance. In the one case, the derived value of (L_a) evidently would be too low and in the other too high.

Method (3) represents a compromise between the two alternative procedures described under method (2), the primary assumption here involved being that the total amount of disturbance between Stations A and B is equal to an amount which, if concentrated at a

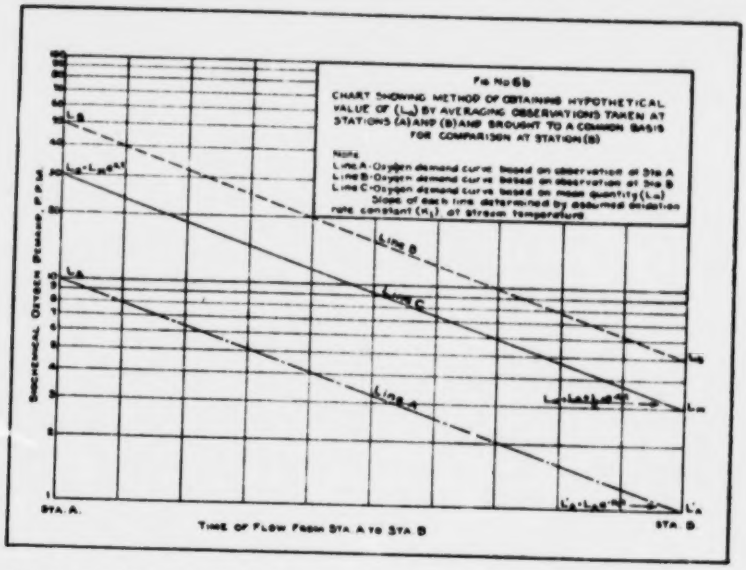


Fig No 6b
CHART SHOWING METHOD OF OBTAINING HYPOTHETICAL VALUE OF (L_m) BY AVERAGING OBSERVATIONS TAKEN AT STATIONS (A) AND (B) AND BROUGHT TO A COMMON BASIS FOR COMPARISON AT STATION (B)
Note
Line A - Oxygen demand curve based on observation at Sta A
Line B - Oxygen demand curve based on observation at Sta B
Line C - Oxygen demand curve based on mean quantity (L_m)
Slope of each line determined by assumed deviation rate constant (K_1) at stream temperature
Line B
Line C
Line A
L_a
L_m
L'_a
$L_a - L_a e^{-K_1 t}$
$L_m - L_m e^{-K_1 t}$
$L'_a - L'_a e^{-K_1 t}$
DISSOLVED OXYGEN PPM
STA. A.
STA. B
TIME OF FLOW FROM STA. A TO STA. B

point immediately below Station A, would account for exactly one-half the observed divergence between (L_B) and (L'_A) at Station B. The effect of this assumption is illustrated in Figure No. 6b by line C, the value (L_m) being an arithmetical mean of (L_B) and (L'_A), or:

$$L_m = \frac{L'_B + L_A e^{-K_1 t}}{2}$$

It will be noted that the total amount of oxygen demand satisfied between Stations A and B along line C is equal to the arithmetical mean of the total amounts satisfied, respectively, along lines A and B; that is:

$$(L_a - L_m) = \frac{(L_A - L'_A) + (L'_B - L_B)}{2}$$

Method (3) thus involves a further assumption that the total amount of oxygen demand satisfied between Stations A and B is equal to the mean of the amounts which would be satisfied under the maximum and minimum possible conditions of disturbance, respectively. In the absence of any specific data as to the actual amount or distribution of such disturbance, an assumption of this nature would appear to be a far more reasonable one than those which are involved in methods (1) and (2). The calculation of (L_a) by this method is simple, its value being given, in terms of the observed quantities (L_A) and (L_B), by the relation:

$$L_a = \left[\frac{L_B + L_A e^{-K_1 t}}{2}\right] e^{K_1 t} - \frac{L_A + L_B e^{K_1 t}}{2}$$

The methods above described have reference solely to the derivation of a value of (L_a) for substitution into formula (1) which, it will be recalled, is based on a primary assumption of an initial oxygen demand, subsequently undisturbed in its orderly rate of decrease through oxidation. It may be argued that procedures based on such an assumption can hardly be applied, logically, to conditions of natural streams in that they fail to take direct account of the manner in which the disturbing influences causing a deviation of (L_B) from (L'_A) are distributed along the stream between two given observation stations. A contention of this kind unquestionably is sound theoretically, but when an attempt is made to develop a procedure based on some definite primary assumption as to the distribution of disturbing influences along a stream, two difficulties present themselves. First is a question as to the kind of distribution which shall be assumed in a given case, with little or no information available on which to base a particular assumption. Second is the obvious fact that for each different assumption made a new and characteristic equation of the type of formula (1) must be developed, starting with a new differential equation of condition, and undergoing the various steps of integration similar to those outlined in Appendix A with respect to formula (1).

The mathematical complexities which such a procedure involves are well illustrated by the example, given in Appendix B, of the derivation of a new resultant oxygen formula based on the most elementary assumption possible regarding distributed disturbance, namely, a uniform distribution of inflow along the stream between Stations A and B. The equation thus derived is as follows:

$$D = \frac{K_1[K_2(1-e^{-K_1 t})\,(L_b - L_a e^{-K_2 t}) - K_1(L_b - L_a e^{-K_1 t})\,(1-e^{-K_2 t})]}{K_2\,(K_2 - K_1)\,(1-e^{-K_1 t})} + D_a e^{-K_2 t}$$

the symbols employed being the same as in formula (1), except that (L_A) and (L_B) represent the observed oxygen demand at Stations A and B, respectively.

Efforts to develop other formulae of the same type, based on other assumptions as to the distribution of inflow, have led to integrated equations even more complex than the one above stated. For purposes of practical application such formulae are so complicated as to be virtually unworkable in most cases. In the absence of specific information as to the manner in which disturbing influence may be distributed in a given instance, the use of a particular formula, based on a single arbitrarily assumed distribution, would be hazardous, moreover, in view of the extreme variability and high degree of uncertainty with which such distributions occur in natural streams. With the admitted theoretical advantages of these procedures over the more simple one embodied in formula (1) their practical disadvantages are of such a nature as to render their use inadvisable for purposes of working calculations.

For use in connection with the analyses of the Ohio River data, to be discussed in the text which follows, the procedure finally adopted as being the most suitable for the purpose at hand was the employment of formula (1), deriving values of (L_a) by averaging those of (L_b) and ($L_A e^{-K_1 t}$), as described under method (3) on pages 23–24. By this comparatively simple procedure values of (L_a) were calculated from the base data given in Table No. 1 for each stretch of the river for which observations were available.

An inspection of formula (1) indicates that a direct solution of it for the value of the reaeration coefficient (K_2) involves mathematical difficulties, since this term appears both as a coefficient and as an exponent. While it is possible that a convenient solution of this equation by means of a nomographic chart might be devised, a fairly simple method in practice has consisted of the indirect procedure of assuming values of (K_2), solving for the corresponding (D) values; plotting these on cross-section paper with (K_2) values as ordinates and (D) values as abscissae, and from a smooth curve drawn through the points, selecting the value of (K_2) corresponding to the known (D) value. The substitution of the value of (K_2), thus obtained, into formula (1) will insure that the proper figure has been chosen from the plot, thus checking the work. This method of procedure is comparatively simple in routine work if the so-called "log-log" slide rule is used, permitting the convenient solution of power functions. It is rarely necessary to obtain more than three points on the smooth curve to obtain the interpolated value of (K_2) sought.

Employing the foregoing method, values of the reaeration coefficient may be readily computed for any river stretch or any condition of flow or season for which the necessary laboratory and hydrometric data are available. From calculations of this kind a set of (K_2) values will be obtained for each particular river stretch

studied, the values varying according to the different conditions of temperature and flow encountered during the period of observations.

The coefficients thus obtained, representing in each case the coincident stream conditions, are modified by three major factors, namely, temperature, stream depth, and turbulence. The temperature influence is, strictly speaking, a compound one, depending upon the resultant effect of temperature variations upon the rate of solution of oxygen at the water surface and upon its velocity of diffusion in the lower strata. These two influences are opposed to each other in the sense that the rate of solution diminishes with rising temperature, while the rate of diffusion increases.

The effect of temperature variations upon the rate of oxygen solution at the surface is governed by the relative amounts of saturation deficit represented by a particular dissolved oxygen content when referred to saturation values at different temperatures. If, for example, the oxygen content be 5 parts per million, its saturation deficit at 20° C. is 4.2 parts and at 30° C. is 2.6 parts, the relative rates of solution at the two temperatures being defined by the proportion 4.2:2.6. Where the oxygen content is expressed, however, in terms of saturation deficit, as is the case in formulas (1) and (3), this temperature relation is taken account of automatically, and the diffusion relationship is the governing influence.

In their New York Harbor studies, Black and Phelps[21] derived experimentally a curve showing the effects of temperature variations upon rates of diffusion of atmospheric oxygen in water. In Figure No. 7 this curve, slightly modified for temperatures below 10° C., is reproduced in such a way as to show relative rates of diffusion at various temperatures with reference to the rate of 20° C. From this curve it appears that the rate of diffusion at 20° C. is doubled at a temperature of about 27.2° C., trebled at 31.4° C., and halved at 10.6° C. It is evident that the temperature of the water, even within ranges ordinarily observed in streams, exerts a great influence upon the rate of diffusion of oxygen in water and hence upon the rate of reaeration of streams. By means of a curve similar to that of Figure No. 7, values of the reaeration coefficients as determined at various prevailing stream temperatures may be reduced to equivalent values at a standard temperature of 20° C. before comparing them with reference to other stream conditions.

The influence of stream depth upon the rate of reaeration is controlled in part by the relation of depth to volume of flow (which, in turn, governs the concentration, in the stream, of oxygen derived

[21] Black, Col. W. M., and Phelphs, E. B., The Discharge of Sewage into New York Harbor. Report made to the Board of Estimate and Apportionment, New York City. 1911.

from reaeration) and probably also in part by the relation of depth to the rate of diffusion. It happens that both the rate of diffusion and volume of flow are power functions of the depth, approximating its square. A glance at the equivalent of (K) in the diffusion formula on page 11, which is given by the quantity $\left(\frac{at\pi^2}{4L^2}\right)$ will show, for example, that the rate of diffusion is inversely proportional to the square of the depth (L). As regards the volume of flow, it is equal to the velocity times the area of flow, the relation

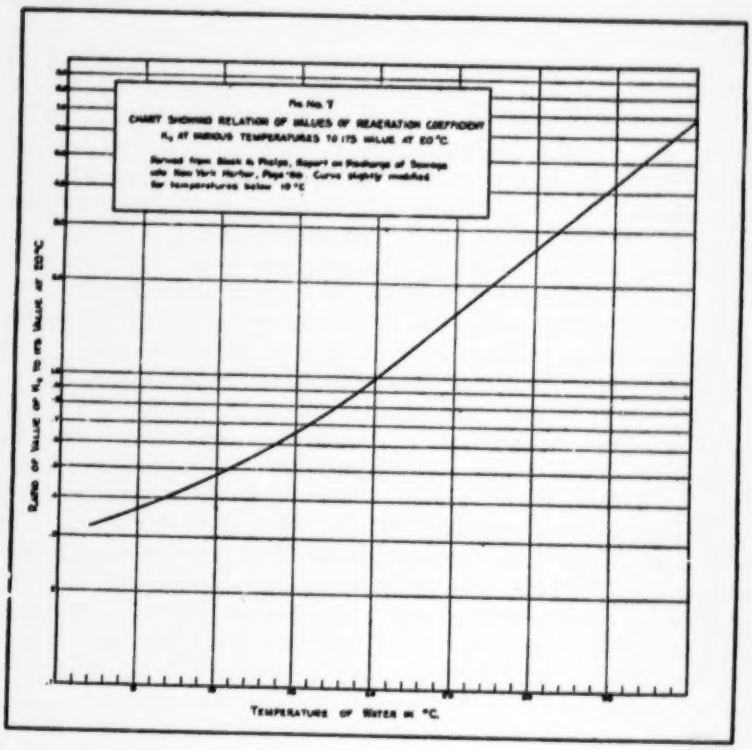


Fig. No. 7
CHART SHOWING RELATION OF VALUES OF REAERATION COEFFICIENT
AT VARIOUS TEMPERATURES TO ITS VALUE AT 20°C
Derived from Streeter & Phelps, Report on Pollution of Sewage
into New York Harbor, Page 166. Curve slightly modified
for temperatures below 10°C
Ratio of Value of K2 to its Value at 20°C
Temperature of Water in °C.

being expressed by the simple hydraulic formula: (Q=AV), in which (Q) denotes the volume of flow, (A) the area, and (V) the velocity. In most streams having a fairly large ratio of width to depth, the area of flow (A) and the velocity (V) are each of them very nearly proportional to the depth (H), so that the relation to volume of flow may be written as being roughly: ($Q=cH^2$). The amount of reaeration per unit of time, or its rate, when measured in terms of oxygen concentration, should be inversely proportional to the quantity of water throughout which the oxygen is distributed;

that is, to the stream discharge (Q). From the relation just shown, this rate should therefore be roughly proportional directly to the square of the depth (H).

As to the relation between reaeration and turbulence, no well-defined theoretical basis exists for judging its nature, the reason being that the turbulence of a stream is an abstract function dependent upon and capable of expression only in terms of some other more concrete stream characteristic. For a given stream type, some definite relationship might reasonably be expected to exist between the turbulence of a stream and its velocity of flow, but it is evident that different types of streams should reveal quite different velocity-turbulence relations. For example, in a stream possessing an even, smooth, deep channel, with flat slopes and either straight or gradually changing direction of flow, the effect of velocity upon turbulence is conceivably much less than in a watercourse having a shallow, rough channel, with steep slopes and sharp changes in direction. Between the two extremes lie numerous well-defined stream types; and a large river such as the Ohio is likely to exhibit throughout its course several of these types.

No very definite theoretical relation exists between turbulence and velocity, even assuming other physical conditions constant. In a general way, turbulence is the result of frictional resistance to flow, and under uniform physical conditions might be expected to be a power function of the velocity of the form:

$$T=cV^n$$

the constants c and n defining the stream type as regards the fixed physical conditions, such as slope, character of bottom, depth, shape, and direction of channel, etc. This subject can therefore be dealt with only empirically.

EXPERIMENTAL

The quantitative working theory of the process of oxygen stream purification which has been developed and expressed in the form of certain algebraic equations is capable of two distinct uses. With all the constants known, it makes possible the computation of actual stream conditions and may even be employed for the determination of future conditions, with an assumed increase in contributing population or an assumed degree of sewage purification. Information is lacking, however, concerning the reaeration or reoxygenation coefficient (K_2) in the formulas, excepting from a purely theoretical standpoint.

It is possible, however, to utilize these formulas for the experimental determinations of this term, by actually measuring the

more readily determinable present stream conditions. To this end analytical data have been obtained in the form of dissolved oxygen and oxygen demand values, and these data will be employed for the determination of the reaeration coefficient. It will then be possible to study this coefficient in connection with the physical stream conditions which affect it and to search for any general relationships which would make the determinations of reaeration a matter of computation for known physical conditions.

It should be noted that the other data necessary for the solution of the condition equation are either known or capable of ready determination in any particular case.

PRESENTATION OF BASE DATA

The scope of the observations constituting the basis of the study embraced daily or thrice-weekly determinations of dissolved oxygen and biochemical oxygen demand of the Ohio River at selected stations throughout its entire length, from Pittsburgh to below Paducah, over a period extending from May 1 to October 15, 1914, supplemented by similar observations in a stretch of the river extending from above Cincinnati to below Louisville, over a period from October 16, 1914, to April 30, 1915. The data for the latter named stretch thus embraced a full year's cycle of stream conditions.

The base data derived from these observations were first reduced to terms of monthly average figures and summarized as shown in Table No. 1, which also includes certain supplementary data employed in the analysis of the laboratory results, such as mean river water temperatures and times of flow between the various sampling stations (columns 2, 3, and 4). The sampling station notations as given in column 1 require explanation. The location of all stations was referred to a point at the junction of the Allegheny and Monongahela Rivers at Pittsburgh, and each station on the Ohio River was given a number corresponding to its location in miles below the reference point. (See Map A, showing the location of the various stations; also Public Health Bulletin No. 143, pp. 98–104, for a description of the stations.) Thus the station designated as "Ohio No. 23" was located 23 miles downstream from the reference point. The tributary stations, designated by name, were located in all except two cases practically at the tributary mouth. The two exceptions were the Allegheny and Monongahela stations, which were located at distances of 7 and 12 miles, respectively, above their junction at Pittsburgh.

In the analysis of the data which follows, 11 stretches of the river have been selected for reaeration study, the various factors involved in the calculations having been transcribed from Table No. 1 and

arranged as shown in Table No. 2. The last two columns in these tables contain values of the reaeration coefficient (K_2) as determined for the actual stream condition by the method outlined on pages 26–27 and as converted to a standard temperature basis of 20° C., using the curve of Figure 7 for the purpose. In Table No. 3 is given a summary of the values of (K_2) thus standardized and brought together from Table No. 2 for reference and comparison.

TABLE No. 1.—*Average results, by months, of laboratory determinations of dissolved oxygen and biochemical oxygen demand at various sampling stations in the Ohio River and at the mouths of designated tributaries, together with collateral data concerning mean river water temperatures and times of flow between successive stations*

MONTH OF MAY, 1914

Sampling station	Mean water temperature, °C.	Mean time of flow, days, from—		Initial dissolved oxygen			Loss during 24 hours' incubation at 20° C. (parts per million)	Total oxygen demand	
		Pittsburgh	Station next above	Parts per million	Per cent saturation	Saturation deficit (parts per million)		Parts per million	Quantity units (parts per million × thousand second-feet)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Allegheny, No. 7	19.9	.....	.....	9.00	98.0	0.19	0.45	2.18	71.5
Monongahela, No. 12	23.5	.....	.....	7.21	83.0	1.39	.36	1.75	16.5
Ohio, No. 3	15.8	0.09	0.09	9.10	91.1	.89	.99	4.80	272.5
Ohio, No. 11	16.0	.30	.21	9.32	93.8	.63	.05	5.10	215.0
Ohio, No. 19	15.8	.48	.18	9.27	92.8	.72	.94	4.56	192.3
Ohio, No. 23	16.0	.56	.08	9.10	91.5	.85	.89	4.32	182.5
Beaver	17.7	.60	.04	9.29	96.8	.31	1.14	5.53	65.8
Ohio, No. 29	14.3	.66	.06	9.53	92.5	.77	1.41	6.84	340.0
Ohio, No. 65	16.6	1.18	.52	8.60	87.6	1.22	.97	4.70	261.6
Ohio, No. 77	16.8	1.36	.20	8.80	90.0	.98	1.07	5.20	289.5
Ohio, No. 88	16.8	1.56	.18	8.89	90.9	.89	1.47	7.13	397.0
Ohio, No. 97	16.3	1.71	.15	8.71	88.1	1.18	1.36	6.60	367.0
Ohio, No. 104	16.4	1.86	.15	8.37	84.9	1.50	1.01	5.05	281.0
Ohio, No. 349	17.0	5.89	4.03	8.43	86.6	1.31	.25	1.21	122.2
Scioto	18.6	6.00	.11	8.26	87.0	1.17	.60	2.92	20.3
Ohio, No. 461	17.6	7.65	1.65	8.20	85.8	1.43	.46	2.24	269.0
Little Miami	18.8	7.67	.02	8.41	89.7	.96	.79	3.84	7.0
Licking	19.4	7.77	.10	8.60	92.6	.68	1.10	5.34	21.4
Ohio, No. 475	17.1	7.87	.10	8.18	84.2	1.54	.79	3.84	483.0
Ohio, No. 482	17.1	8.00	.13	8.17	84.2	1.55	.85	4.13	520.0
Ohio, No. 488	17.0	8.11	.10	8.14	83.5	1.60	.64	3.11	392.0
Miami	17.5	9.12	.02	8.63	89.5	1.01	.98	4.76	21.0
Ohio, No. 492	16.9	8.16	.04	8.09	82.9	1.67	.78	3.79	494.0
Ohio, No. 598	17.6	10.14	1.98	8.01	83.3	1.61	.70	3.40	460.0
Ohio, No. 611	18.0	10.42	.28	7.90	82.8	1.64	.73	3.55	521.0

MONTH OF JUNE, 1914

Allegheny, No. 7	23.0	.....	.....	7.59	87.5	1.09	0.31	1.51	9.9
Monongahela, No. 12	24.0	.....	.....	6.41	75.2	2.09	.27	1.31	3.7
Ohio, No. 3	24.0	0.38	0.38	5.40	63.3	3.13	.34	1.65	15.5
Ohio, No. 11	25.0	1.45	1.07	7.29	87.0	1.09	.43	2.08	19.5
Ohio, No. 19	24.0	2.22	.77	7.39	86.6	1.14	.38	1.85	17.4
Ohio, No. 23	23.4	2.54	.32	7.29	84.7	1.33	.32	1.55	14.5
Beaver	23.4	2.70	.16	7.70	89.3	.92	.49	2.38	2.0
Ohio, No. 29	.....	2.96	.26	8.01	93.0	.66	.18	.87	5.8
Ohio, No. 65	22.7	4.44	1.48	7.79	89.4	.93	.52	2.52	26.0
Ohio, No. 77	23.7	5.46	1.02	8.11	94.8	.46	.57	2.77	28.5
Ohio, No. 88	23.7	6.11	.65	8.18	95.6	.39	.75	3.64	37.5
Ohio, No. 97	22.6	6.69	.58	7.71	88.2	1.03	.57	2.77	28.5
Ohio, No. 104	23.3	7.23	.54	7.67	88.9	.95	.71	3.45	35.5

TABLE No. 1.—*Average results, by months, etc.*—Continued

MONTH OF JUNE, 1914—Continued

Sampling station	Mean water temperature, °C	Mean time of flow, days, from— Pittsburgh	Mean time of flow, days, from— Station next above	Initial dissolved oxygen: Parts per million	Initial dissolved oxygen: Per cent saturation	Initial dissolved oxygen: Saturation deficit (parts per million)	Loss during 24 hours' incubation at 20° C. (parts per million)	Total oxygen demand: Parts per million	Total oxygen demand: Quantity units (parts per million × thousand second-feet)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Ohio, No. 349	25.5	17.32	10.09	7.42	89.4	.88	.42	2.04	36.5
Scioto	24.5	17.52	.20	7.76	90.6	.69	1.05	5.00	5.0
Ohio, No. 461	26.2	20.24	2.72	7.88	96.3	.21	.50	2.43	48.1
Licking	27.6	20.54	.30	6.88	86.2	1.10	.98	4.76	4.9
Ohio, No. 475	26.2	20.91	.37	6.35	77.5	1.84	.90	4.37	91.6
Ohio, No. 482	25.8	21.39	.48	6.79	82.3	1.46	.91	4.42	92.8
Ohio, No. 498	25.6	21.64	.25	6.45	77.9	1.83	.74	3.60	75.5
Miami	25.6	21.69	.05	7.90	95.4	.36	1.50	7.28	9.0
Ohio, No. 492	25.0	21.77	.08	6.48	77.3	1.90	.90	4.37	98.0
Ohio, No. 508	25.3	27.83	6.06	7.51	90.2	.82	.95	4.61	132.8
Ohio, No. 611	25.9	28.78	.95	7.38	89.6	.86	.82	3.98	114.5
Ohio, No. 904	26.9	41.62	12.79	7.24	89.6	.94	.40	1.94	99.4
Cumberland	29.4	41.96	.34	5.56	68.3	2.15	.00	.00	.0
Ohio, No. 920	26.9	42.34	.38	7.44	92.1	.64	.35	1.70	86.9
Tennessee	29.6	42.40	.06	6.56	85.3	1.13	.28	1.26	21.5
Ohio, No. 933	28.0	42.95	.55	7.20	90.9	.72	.77	3.74	255.0

MONTH OF JULY, 1914

Allegheny, No. 7	24.5			7.36	87.1	1.02	0.33	1.60	5.5
Monongahela, No. 12	23.0			6.58	78.6	1.78	.43	2.09	4.0
Ohio, No. 3	25.1	0.68	0.68	4.36	52.1	4.00	.44	2.16	11.5
Ohio, No. 11	24.0	2.49	1.81	7.21	84.5	1.32	.41	1.99	10.7
Ohio, No. 19	24.2	3.85	1.36	7.46	87.6	1.04	.44	2.14	11.5
Ohio, No. 23	24.5	4.45	.60	7.21	85.3	1.24	.48	2.33	12.6
Beaver	23.8	4.74	.29	7.43	86.7	1.14	.64	3.11	1.4
Ohio, No. 65	24.6	7.45	2.71	7.80	88.9	.94	.24	1.16	7.7
Ohio, No. 77	25.0	9.09	1.73	7.84	93.5	.54	.39	1.90	12.4
Ohio, No. 88	25.0	10.20	1.01	7.86	93.8	.52	.49	2.38	15.8
Ohio, No. 97	24.8	11.20	1.00	7.74	92.1	.67	.33	1.60	11.6
Ohio, No. 104	24.8	12.11	.91	7.49	89.0	.92	.31	1.51	10.0
Ohio, No. 349	26.7	24.90	12.79	6.83	84.2	1.28	.38	1.85	30.3
Scioto	25.6	25.10	.20	7.57	89.6	.71	.90	4.37	3.2
Ohio, No. 461	27.1	27.86	2.76	7.70	93.6	.35	.40	1.94	36.3
Licking	29.3	28.21	.35	6.63	85.7	1.10	.83	4.03	.4
Ohio, No. 475	27.3	28.64	.43	6.00	74.9	2.02	.81	3.94	74.8
Ohio, No. 482	26.8	29.20	.56	6.76	83.5	1.34	.81	3.93	74.6
Ohio, No. 488	27.2	29.46	.26	6.62	83.3	1.42	.68	3.30	63.7
Ohio, No. 492	26.5	29.60	.14	6.43	77.8	1.71	.66	3.20	64.0
Ohio, No. 508	27.0	36.86	7.26	7.70	95.5	.37	1.01	4.90	113.0
Ohio, No. 611	27.2	38.06	1.20	7.45	92.6	.59	.84	4.08	94.4
Ohio, No. 619	27.2	38.38	.32	7.06	87.8	.98	.98	4.76	109.8
Ohio, No. 904	28.3	53.19	14.81	7.51	95.5	.36	.32	1.55	47.7
Cumberland	26.9	53.64	.45	6.28	80.6	1.50	.36	1.26	10.1
Ohio, No. 920	28.9	54.05	.41	7.34	94.4	.44	.21	1.04	40.3
Tennessee	29.7	54.11	.06	6.47	84.3	1.20	.18	.87	16.7
Ohio, No. 933	29.0	54.60	.50	6.90	88.8	.87	.25	1.36	78.8

TABLE No. 1.—*Average results, by months, etc.*—Continued

MONTH OF AUGUST, 1914

Sampling station	Mean water temperature, °C.	Mean time of flow, days, from—		Initial dissolved oxygen			Loss during 24 hours' incubation at 20° C. (parts per million)	Total oxygen demand	
		Pittsburgh	Station next above	Parts per million	Per cent saturation	Saturation deficit (parts per million)		Parts per million	Quantity units (parts per million × thousand second-feet)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Allegheny No. 7	23.0	.....	.....	7.16	82.4	1.52	0.39	1.90	3.1
Monongahela No. 12	24.8	.....	.....	6.40	76.0	2.01	.71	3.45	7.1
Ohio No. 3	23.0	1.00	1.00	2.55	34.0	5.53	.61	2.96	11.0
Ohio No. 11	24.5	3.73	3.73	6.50	78.9	1.95	.51	2.48	9.2
Ohio No. 19	24.4	5.71	1.98	6.87	81.1	1.60	.52	2.52	9.4
Ohio No. 23	24.0	6.59	.88	6.55	77.9	1.88	.35	1.85	6.9
Beaver	23.0	7.01	.42	7.40	85.3	1.28	.52	2.53	.9
Ohio No. 65	24.2	11.49	4.48	7.75	91.2	.73	.28	1.36	6.3
Ohio No. 77	24.6	13.81	2.47	7.95	94.2	.49	.12	.58	2.7
Ohio No. 86	24.4	15.47	1.51	7.76	91.7	.71	.60	2.43	11.3
Ohio No. 97	24.2	16.93	1.45	7.60	93.0	.50	.55	2.67	12.4
Ohio No. 104	23.9	18.26	1.34	7.59	89.9	.96	.35	1.70	7.9
Ohio No. 349	23.7	34.88	16.62	7.09	83.8	1.18	.37	1.80	21.2
Scioto	24.8	35.12	.24	7.37	87.3	1.04	.83	4.03	3.9
Ohio No. 461	26.4	38.34	3.22	7.27	89.1	.89	.49	2.38	33.0
Licking	27.2	38.64	.30	6.58	81.8	1.46	1.01	4.90	3.6
Ohio No. 475	26.3	39.22	.58	4.97	60.8	3.20	.79	3.84	58.4
Ohio No. 482	25.8	40.01	.69	6.39	77.5	1.86	.74	3.59	53.0
Ohio No. 488	25.9	40.30	.29	6.24	75.8	2.00	.65	3.15	47.9
Ohio No. 492	25.0	40.45	.15	6.23	74.3	2.15	.68	3.30	52.8
Ohio No. 508	26.0	48.98	8.67	7.81	96.6	.41	.33	1.60	29.3
Ohio No. 611	27.0	50.42	1.44	7.79	95.0	.28	.38	1.85	33.8
Ohio No. 619	27.1	50.80	.38	7.33	91.1	.73	.60	2.91	53.2
Ohio No. 904	27.5	67.54	16.74	7.18	89.9	.81	.16	.78	20.6
Cumberland	28.0	67.88	.34	6.28	79.3	1.64	.18	.87	6.0
Ohio No. 920	27.9	68.37	.49	7.78	96.2	.15	.40	1.94	64.5
Tennessee	28.4	68.43	.06	6.95	88.4	.91	.24	1.17	19.5
Ohio No. 933	27.9	69.00	.57	7.09	89.4	.84	.30	1.46	72.0

MONTH OF SEPTEMBER, 1914

Allegheny No. 7	19.0	.....	.....	8.25	88.2	1.10	0.46	2.24	5.0
Monongahela No. 12	21.0	.....	.....	6.84	71.8	2.45	.55	2.67	3.0
Ohio No. 3	21.0	1.06	1.05	2.71	30.2	6.26	.59	2.86	9.6
Ohio No. 11	20.5	4.11	3.06	6.59	72.6	2.49	.44	2.14	7.2
Ohio No. 19	20.5	6.41	2.30	7.09	78.0	1.99	.54	2.62	8.8
Ohio No. 23	20.3	7.35	.94	5.35	58.7	3.77	.08	.39	1.3
Beaver	19.0	7.82	.47	7.73	82.7	1.38	.27	1.31	.5
Ohio No. 65	19.8	13.49	5.67	8.45	91.6	.76	.23	1.12	5.0
Ohio No. 77	20.3	15.88	2.54	8.68	95.2	.44	.32	1.55	7.0
Ohio No. 86	20.4	17.54	1.51	8.77	96.3	.33	.56	2.82	12.7
Ohio No. 97	20.4	19.02	1.45	8.72	95.9	.38	.56	2.72	12.2
Ohio No. 104	20.3	20.36	1.34	8.32	91.3	.80	.53	2.56	11.6
Ohio No. 349	22.3	36.12	15.76	7.74	88.0	1.04	.30	1.46	20.0
Scioto	21.1	36.34	.22	8.29	92.3	.68	.79	3.84	3.1
Ohio No. 461	22.1	39.27	2.93	7.64	85.8	1.17	.44	2.14	36.6
Licking	22.5	39.65	.38	7.43	84.9	1.32	.73	3.55	.8
Ohio No. 475	22.6	40.11	.46	5.87	67.1	2.87	.80	4.32	78.5
Ohio No. 482	22.5	40.72	.61	6.91	79.0	1.84	.74	3.59	62.8
Ohio No. 488	22.3	40.99	.27	6.76	77.0	2.02	.62	3.01	52.7
Ohio No. 492	22.0	41.14	.15	6.82	77.2	1.91	.64	3.11	56.9
Ohio No. 508	22.9	48.86	7.72	8.43	97.1	.26	.52	2.52	52.6
Ohio No. 611	22.9	50.14	1.28	8.42	97.0	.27	.56	2.72	56.8
Ohio No. 619	22.6	50.48	.34	8.21	93.9	.53	.69	3.35	70.0
Ohio No. 904	23.3	65.73	15.25	7.31	84.7	1.32	.25	1.21	45.1
Cumberland	24.5	66.11	.38	6.77	80.1	1.68	.32	1.55	10.6
Ohio No. 920	23.7	66.46	.37	7.46	87.0	1.11	.43	2.09	92.5
Tennessee	25.0	66.53	.05	7.53	89.8	.85	.16	.78	12.4
Ohio No. 933	24.1	67.11	.56	7.18	84.3	1.33	.30	1.46	67.6

TABLE No. 1.—*Average results, by months, etc.*—Continued

OCTOBER 1–15, 1914

Sampling station	Mean water temperature, °C	Mean time of flow, days, from—		Initial dissolved oxygen			Loss during 24 hours' incubation at 20° C. (parts per million)	Total oxygen demand	
		Pittsburgh	Station next above	Parts per million	Per cent saturation	Saturation deficit (parts per million)		Parts per million	Quantity units (parts per million × thousand second-feet)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Allegheny No. 7	17.0	.....	.....	8.63	88.7	1.11	0.23	1.12	0.9
Monongahela No. 12	19.7	.....	.....	5.97	64.8	3.25	.35	1.70	1.4
Ohio No. 3	19.0	2.11	2.11	.72	7.2	8.63	.29	1.41	2.3
Ohio No. 11	18.3	8.34	6.23	6.37	67.2	3.11	.27	1.31	2.2
Ohio No. 19	18.8	12.71	4.37	6.79	72.3	2.60	.19	.92	1.5
Ohio No. 23	17.9	14.60	1.89	6.73	70.4	2.83	.11	.53	.9
Beaver	16.0	15.54	.94	7.94	79.8	1.62	.42	2.04	.5
Ohio No. 65	17.7	25.95	10.41	9.00	93.8	.50	.44	2.14	5.3
Ohio No. 77	18.2	30.69	5.02	8.70	91.6	.80	.33	1.60	4.0
Ohio No. 88	18.6	33.70	2.73	8.45	89.5	.98	.24	1.65	4.1
Ohio No. 97	18.2	36.36	2.66	8.48	89.2	1.02	.38	1.85	4.6
Ohio No. 104	18.1	38.74	2.38	7.83	82.3	1.69	.29	1.41	3.5
Ohio No. 349	20.2	68.38	29.64	8.17	89.5	.98	.38	1.85	8.2
Scioto	19.4	68.83	.45	8.68	93.	.60	.78	3.79	1.9
Ohio No. 461	20.1	73.25	4.42	8.88	97.2	.27	.37	1.80	13.0
Licking	19.3	74.12	.87	8.04	86.5	1.26	.79	3.84	3.7
Ohio No. 475	20.6	75.07	.95	4.75	52.5	4.31	1.32	6.41	54.0
Ohio No. 482	18.5	76.27	1.20	6.82	72.2	2.62	.84	4.08	34.4
Ohio No. 488	20.1	76.69	.42	6.82	76.0	2.33	.82	3.98	33.5
Ohio No. 492	20.0	76.88	.19	7.03	76.7	2.14	.80	3.88	35.2
Ohio No. 598	19.6	94.68	17.80	9.38	101.5	—.14	.51	2.48	25.8
Ohio No. 611	20.8	98.01	3.33	9.38	103.9	—.35	.73	3.55	37.0
Ohio No. 619	20.8	98.71	.70	9.01	99.8	.02	.73	3.55	37.0
Ohio No. 904	21.3	127.7	29.00	8.31	93.0	.63	.42	2.04	46.3
Cumberland	21.5	128.2	.48	7.17	80.4	.74	.47	2.28	7.8
Ohio No. 920	21.2	128.6	.42	8.34	93.0	.63	.44	2.14	57.8
Tennessee	21.6	128.7	.06	8.35	94.0	.54	.37	1.80	22.1
Ohio No. 933	21.4	129.4	.71	8.03	89.8	.90	.31	1.51	59.4
MONTH OF OCTOBER, 1914									
Ohio No. 461	18.0	56.72	.....	9.00	94.3	0.54	0.66	3.21	40.7
Licking	16.6	57.19	0.47	8.88	90.5	.94	1.05	5.10	18.4
Ohio No. 475	17.8	57.65	.46	6.67	69.6	2.91	1.57	7.62	128.8
Ohio No. 482	17.9	58.35	.70	7.88	82.5	1.68	1.06	5.15	87.0
Ohio No. 488	17.4	58.64	.29	7.79	80.6	1.67	1.04	5.05	85.3
Ohio No. 492	17.1	58.79	.15	8.36	86.1	1.36	1.21	5.87	106.3
Ohio No. 598	18.0	67.40	8.61	8.73	91.6	.81	.44	2.14	47.9
Ohio No. 611	18.1	68.73	1.33	8.92	93.8	.65	.54	2.62	[illegible]
Ohio No. 619	18.0	69.08	.35	8.64	90.6	.90	.57	2.77	62.0
MONTH OF NOVEMBER, 1914									
Ohio No. 461	7.7	37.20	.....	11.94	99.8	0.02	1.90	9.23	105.2
Licking	7.6	37.74	0.54	12.72	106.1	—.73	2.20	10.69	2.5
Ohio No. 475	8.3	38.28	.54	10.69	90.7	1.10	2.44	11.85	139.8
Ohio No. 482	8.3	39.11	.83	11.41	96.8	.38	2.09	10.15	119.8
Ohio No. 488	8.6	39.43	.32	11.08	94.3	.67	1.98	9.62	113.5
Ohio No. 492	8.5	39.60	.17	10.50	89.5	1.23	2.10	10.40	131.0
Ohio No. 598	10.0	50.04	10.44	11.46	101.1	—.13	1.47	7.14	90.6
Ohio No. 611	9.9	52.08	2.04	11.75	103.4	—.39	1.63	.....	.....
Ohio No. 619	10.2	52.60	.52	11.16	99.6	.12	1.46	7.09	87.1

TABLE No. 1.—*Average results, by months, etc.*—Continued

MONTH OF DECEMBER, 1914

Sampling station	Mean water temperature, °C.	Mean time of flow, days, from—		Initial dissolved oxygen			Loss during 24 hours' incubation at 20° C. (parts per million)	Total oxygen demand	
		Pittsburgh	Station next above	Parts per million	Per cent saturation	Saturation deficit (parts per million)		Parts per million	Quantity units (parts per million × thousand second-feet)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Ohio No. 461	4.7	8.69	.....	11.07	79.7	2.83	1.56	7.58	622.0
Licking	.....	8.82	0.13	10.26	.....	.....	2.44	11.85	72.2
Ohio No. 475	4.7	9.96	.14	11.38	81.9	2.52	2.50	12.14	1,070.0
Ohio No. 482	4.6	9.13	.17	11.50	82.6	2.43	2.48	12.05	1,063.0
Ohio No. 488	4.9	9.25	.12	11.39	82.4	2.44	2.22	10.78	960.0
Miami	5.2	9.27	.02	12.92	99.0	.14	2.46	11.95	25.8
Ohio No. 492	4.8	9.33	.06	11.49	82.8	2.44	2.51	12.20	1,105.0
Ohio No. 598	4.9	11.93	2.60	11.78	89.7	1.35	1.70	8.25	796.0
Ohio No. 611	3.7	12.29	.36	11.84	89.5	1.40	2.08	10.00	965.0
Ohio No. 619	3.6	12.44	.15	12.20	91.9	1.07	2.15	10.45	1,008.0

MONTH OF JANUARY, 1915

Ohio No. 461	1.2	6.42	.....	12.78	99.3	1.37	1.46	7.09	1,219.0
Licking	.....	6.52	0.10	12.60	.....	.....	1.24	6.02	55.8
Ohio No. 475	1.5	6.59	.07	13.30	94.8	.73	2.12	10.30	1,893.0
Ohio No. 482	1.5	6.71	.12	13.26	94.5	.77	1.97	9.56	1,756.0
Ohio No. 488	1.4	6.79	.09	13.18	93.9	.89	1.84	8.92	1,640.0
Miami	1.0	6.81	.02	13.33	93.8	.90	2.08	10.02	39.0
Ohio No. 492	1.3	6.83	.02	13.11	93.0	1.00	1.91	9.27	1,740.0
Ohio No. 598	1.6	8.46	1.59	12.80	91.4	1.20	1.89	9.18	2,036.0
Ohio No. 611	1.5	8.66	.22	12.74	90.8	1.29	1.99	9.65	2,140.0
Ohio No. 619	1.6	8.75	.11	12.67	90.5	1.33	2.02	9.80	2,076.0

MONTH OF FEBRUARY, 1915

Ohio No. 461	2.7	6.23	.....	12.36	91.0	1.23	1.09	5.30	1,181.0
Little Miami	3.4	6.24	0.01	12.16	91.0	1.18	1.10	5.34	61.4
Licking	.....	6.32	.08	11.64	.....	.....	.64	3.11	33.1
Ohio No. 475	3.3	6.41	.11	12.78	95.5	.60	1.60	7.76	1,900.0
Ohio No. 482	3.2	6.51	.10	12.76	95.1	.65	1.48	7.19	1,760.0
Ohio No. 488	3.2	6.59	.08	12.73	95.0	.68	1.46	7.08	1,735.0
Miami	4.1	6.61	.02	12.38	94.5	.72	1.58	7.67	136.1
Ohio No. 492	3.2	6.64	.13	12.65	94.3	.76	1.57	7.62	2,004.0
Ohio No. 598	3.0	8.07	1.43	12.41	92.2	1.07	1.61	7.81	2,412.0
Ohio No. 611	3.3	8.27	.20	12.51	93.7	.87	1.48	7.19	2,220.0
Ohio No. 619	3.3	8.37	.10	12.47	93.3	.91	1.44	6.99	2,158.0

MONTH OF MARCH, 1915

Ohio No. 461	3.9	9.26	.....	11.99	91.1	1.17	0.98	4.76	378.0
Little Miami	4.2	9.28	0.02	12.11	92.8	.95	1.02	4.95	3.1
Licking	.....	9.39	.11	12.00	.....	.....	.80	3.88	16.3
Ohio No. 475	4.5	9.53	.14	12.33	92.3	.64	1.48	7.18	604.0
Ohio No. 482	4.5	9.69	.16	12.33	91.9	.64	1.29	6.26	527.0
Ohio No. 488	4.5	9.81	.12	12.29	94.7	.66	1.29	6.26	527.0
Miami	5.2	9.83	.02	11.71	92.0	1.03	1.22	5.92	17.9
Ohio No. 492	4.4	9.89	.06	12.19	93.8	.81	1.38	6.70	584.0
Ohio No. 598	4.7	12.45	2.56	12.21	94.7	1.69	1.41	6.85	676.0
Ohio No. 611	4.9	12.80	.35	12.26	95.8	1.57	1.27	6.17	609.0
Ohio No. 619	4.9	12.95	.15	12.22	95.2	.61	1.15	5.58	551.0

TABLE No. 1.—*Average results, by months, etc.*—Continued

MONTH OF APRIL, 1915

Sampling station	Mean water temperature, °C.	Mean time of flow, days, from—		Initial dissolved oxygen			Loss during 24 hours' incubation at 20° C. (parts per million)	Total oxygen demand	
		Pittsburgh	Station next above	Parts per million	Per cent saturation	Saturation deficit (parts per million)		Parts per million	Quantity units (parts per million × thousand second-feet)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Ohio No. 461	12.3	12.46	.....	10.00	93.0	0.76	0.58	2.82	112.9
Little Miami	14.4	12.49	0.03	9.92	96.5	.36	.68	3.30	.9
Licking	.....	12.65	.16	9.82	.....	.....	.60	2.92	4.0
Ohio No. 475	12.4	12.84	.19	10.19	94.8	.55	1.18	5.72	238.0
Ohio No. 482	12.3	13.10	.26	10.20	94.8	.56	.98	4.76	198.0
Ohio No. 488	12.0	13.27	.17	10.14	93.7	.69	.98	4.76	198.0
Miami	14.4	13.30	.03	10.60	103.1	−.32	1.40	6.80	12.9
Ohio No. 492	11.9	13.37	.07	10.05	92.6	.81	1.18	5.73	249.0
Ohio No. 598	12.8	17.53	4.16	10.41	97.8	.24	.79	3.84	185.3
Ohio No. 611	12.8	18.25	.72	10.83	101.7	−.18	1.03	5.00	241.5
Ohio No. 619	12.8	18.46	.21	10.66	100.3	−.03	1.09	5.29	255.5

TABLE No. 2.—*Factors in resultant oxygen formula (1) obtained from data in Table No. 1 (by months)*

Notation:
D_A = Observed dissolved oxygen content of the river at the upper station, in terms of parts per million of deficit below the oxygen saturation value at the mean river water temperature between the two stations.
D_B = Observed dissolved oxygen content of the river at the lower station, in the same terms as (D_A).
L_a = Corrected initial oxygen demand at the upper station, in terms of parts per million of oxygen.
t = Mean time of flow from the upper to the lower station, in days.
T = Observed mean temperature of the river water, in degrees Centigrade.
K_1 = Coefficient of deoxygenation at the observed mean river water temperature. (Calculated from a value, 0.1 at 20° C.)
K_2 = Coefficient of reaeration, calculated from the other terms.

Month	At temperature of river water							K_2 (at 20° C.)	Mean velocity of flow (feet per second)
	D_A (parts per million)	D_B (parts per million)	L_a (parts per million)	t (days)	T (° C.)	K_1	K_2		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Station 5 to station 11:									
May 1914	0.89	0.63	2.54	0.21	15.9	0.083	0.990	1.38	2.48
June	3.13	1.09	1.91	1.07	24.5	.123	.544	.35	.50
July	4.00	1.32	2.73	1.81	24.6	.124	.380	.25	.29
August	5.53	1.95	4.22	2.73	24.8	.125	.276	.17	.19
September	6.28	2.49	3.59	3.06	20.8	.104	.197	.18	.18
Oct. 1–15	8.63	3.11	3.30	6.23	18.7	.094	.105	.12	.09
Station 11 to station 19:									
May, 1914	.63	.72	4.89	.18	15.9	.083	.245	.34	2.50
June	1.09	1.14	2.20	.77	24.5	.123	.192	.12	.59
July	1.32	1.04	2.54	1.36	24.1	.121	.289	.20	.32
August	1.95	1.60	3.44	1.98	24.4	.122	.217	.14	.23
September	2.49	1.39	3.14	2.30	20.5	.102	.144	.14	.19
Oct. 1–15	3.11	.60	1.74	4.37	18.6	.092	.052	.06	.10
Station 23 to station 65:									
May, 1914	.85	1.22	5.16	.62	16.3	.084	.200	.30	4.20
June	1.33	.93	2.94	1.90	22.9	.115	.330	.25	1.36
July	1.24	.94	2.65	3.00	24.8	.124	.230	.15	.86
August	1.88	.75	3.69	4.90	24.4	.121	.260	.17	.83
September	3.77	.76	2.56	6.14	20.0	.100	.190	.19	.43
Oct. 1–15	2.83	.50	15.74	11.35	18.0	.091	.353	.43	.23
Station 65 to station 77:									
May, 1914	1.21	.72	4.89	.18	15.9	.083	.245	1.11	3.64
June	.93	.46	3.08	1.02	23.2	.116	.801	.89	.71
July	.94	.84	2.14	1.73	24.8	.125	.436	.26	.44
August	.75	.49	1.26	2.47	24.4	.122	.253	.17	.31
September	.76	.44	1.92	2.54	20.0	.100	.334	.33	.31
Oct. 1–15	.50	.50	3.34	5.02	18.0	.091	.189	.23	.18

TABLE No. 2.—*Factors in resultant oxygen formula, etc.*—Continued

Month	At temperature of river water							K_2 (at 20° C.)	Mean velocity of flow (feet per second)
	D_A (parts per million)	D_B (parts per million)	L_a (parts per million)	t (days)	T (° C.)	K_1	K_3		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Station 77 to station 88:									
May, 1914	0.98	0.89	6.20	0.18	16.6	0.086	0.802	1.07	3.68
June	.46	.39	3.55	.65	23.7	.118	1.005	.71	1.01
July	.54	.52	2.85	1.01	25.0	.126	.629	.33	.62
August	.49	.71	2.15	1.51	24.5	.123	.345	.16	.40
September	.44	.33	2.87	1.51	20.4	.102	.716	.70	.49
Oct. 1-15	.80	.98	2.29	2.73	18.4	.093	.144	.16	.22
Station 104 to station 349:									
May, 1914	1.50	1.31	3.80	4.03	16.7	.086	.180	.24	3.72
June	.96	.88	19.1	10.09	24.4	.124	.388	.18	1.48
July	.92	1.28	41.6	12.79	25.8	.131	.302	.12	1.18
August	.95	1.18	115.8	16.62	24.8	.125	.212	.14	.91
September	.80	1.04	33.3	15.78	21.3	.108	.172	.15	.98
Oct. 1-15	1.69	.98	618.2	20.64	19.2	.097	.200	.22	.51
Station 349 to station 461:									
May, 1914	1.31	1.42	2.66	1.78	17.3	.090	.128	.16	3.90
June	.88	.21	3.90	2.92	23.8	.130	1.140	.65	2.36
July	1.28	.35	2.44	2.96	26.9	.142	.530	.28	2.32
August	1.18	.89	3.11	3.46	26.0	.137	.258	.15	1.98
September	1.04	1.17	2.84	3.15	22.2	.111	.162	.13	2.18
Oct. 1-15	.96	.54	4.08	4.87	20.2	.102	.340	.33	1.41
Station 475 to station 482:									
May, 1914	1.54	1.55	4.02	.13	17.1	.088	.225	.29	3.35
June	1.84	1.46	4.77	.48	26.0	.132	.570	.32	.93
July	2.02	1.34	4.21	.56	27.0	.139	.650	.33	.79
August	3.20	1.88	4.20	.59	26.0	.132	.460	.26	.65
September	2.87	1.84	4.26	.61	22.6	.113	.514	.40	.73
October	2.91	1.68	6.75	.70	17.8	.090	.601	.73	.63
November	1.10	.38	11.60	.83	8.3	.056	1.750	3.98	.54
December	2.52	2.43	12.24	.17	4.6	.069	.345	.96	2.69
January, 1915	.73	.77	10.00	.12	1.5	.043	.365	1.14	3.83
February	.60	.45	7.50	.10	3.2	.046	.202	.61	4.29
March	.64	.64	6.76	.16	4.5	.049	.518	1.43	2.74
April	.55	.56	5.35	.26	12.4	.071	.612	1.09	1.73
Station 482 to station 488:									
May, 1914	1.55	1.00	3.71	.10	17.0	.087	.040	.05	3.61
June	1.46	1.83	4.08	.25	25.7	.130			1.47
July	1.34	1.42	3.76	.26	27.0	.138	.256	.13	1.40
August	1.86	2.00	3.47	.29	26.1	.136	.118	.07	1.23
September	1.84	2.03	3.37	.27	22.4	.112	.060	.05	1.34
October	1.68	1.67	5.20	.29	17.6	.090	.286	.35	1.23
November	.38	.67	10.10	.32	8.4	.059	.358	.81	1.12
December	2.43	2.44	11.63	.12	4.8	.050	.217	.39	3.00
January, 1915	.77	.89	9.35	.09	1.4	.043	.053	.10	4.11
February	.65	.68	7.13	.08	3.2	.046	.265	.78	4.57
March	.64	.68	6.29	.12	4.5	.049	.265	.71	3.00
April	.56	.69	4.87	.17	12.2	.070	.075	.14	2.12
Station 492 to station 598:									
May, 1914	1.67	1.61	4.74	1.98	17.2	.089	.213	.27	3.26
June	1.90	.82	15.38	6.06	25.2	.126	.532	.32	1.07
July	.71	.37	24.80	7.26	26.6	.135	1.090	.57	.89
August	2.15	.41	27.30	8.67	23.5	.128	.788	.46	.74
September	1.01	.26	11.08	7.72	22.4	.112	.761	.61	.84
October	1.36	.81	9.18	8.61	17.6	.090	.247	.31	.73
November [1]									
December	2.44	1.35	11.60	2.60	4.3	.049	.380	1.05	2.48
January, 1915	1.60	1.20	10.80	1.79	1.4	.043	.307	.99	4.06
February	.76	1.07	8.41	1.43	3.1	.047	.200	.59	4.51
March	1.38	1.41	7.98	2.56	4.6	.050	.234	.65	2.52
April	.81	.24	6.09	4.16	12.4	.071	1.050	1.88	1.56
Station 598 to station 611:									
May, 1914	1.61	1.64	3.39	.28	17.8	.090	.170	.21	2.94
June	.82	.88	4.97	.98	25.6	.129	.613	.36	.85
July	.37	.69	5.41	1.30	27.1	.139	.925	.47	.68
August	.41	.28	2.22	1.44	26.5	.135	.792	.43	.66
September	.28	.27	3.17	1.28	22.9	.114	1.140	.87	.63
October	.81	.60	4.68	2.33	18.0	.091	.446	.54	.60
November	−.13	−.39	5.90	2.04	10.0	.063			
December	1.35	1.40	9.20	.36	3.8	.047	.264	.75	2.24
January, 1915	1.20	1.29	9.57	.21	1.6	.043	.185	.58	3.73
February	1.07	.87	7.58	.19	3.2	.046	.630	2.44	4.11
March	1.69	1.57	6.94	.35	4.8	.050	.295	.82	2.28
April	.24	−.18	4.78	.72	12.8	.074			

[1] Results abnormal; omitted.

TABLE No. 8.—*Derived values of* (K_2) *at 20° C.* (*from Table No. 2*)

Stretch	May (1914)	June	July	August	September	Oct. 1-15	October	November	December	January (1915)	February	March	April
3-11	1.38	0.35	0.25	0.17	0.18	0.12	.....	.....	.....	.....	.....	.....	.....
11-19	.34	.12	.20	.14	.14	.06	.....	.....	.....	.....	.....	.....	.....
23-65	.30	.25	.15	.17	.19	.45	.....	.....	.....	.....	.....	.....	.....
65-77	1.11	.59	.28	.17	.33	.23	.....	.....	.....	.....	.....	.....	.....
77-88	1.07	.71	.33	.16	.70	.16	.....	.....	.....	.....	.....	.....	.....
104-349	.24	.18	.12	.14	.15	.22	.....	.....	.....	.....	.....	.....	.....
349-461	.16	.65	.28	.15	.13	.33	.....	.....	.....	.....	.....	.....	.....
475-482	.29	.32	.33	.26	.40	.....	0.73	3.98	0.96	1.14	0.61	1.43	1.09
482-488	.05	.....	.13	.07	.05	.....	.35	.81	.39	.10	.78	.71	.14
492-598	.27	.32	.57	.46	.61	.....	.31	.....	1.05	.99	.59	.65	1.88
598-611	.21	.36	.47	.43	.87	.....	.54	.....	.75	.58	2.44	.82	.....

THE OXYGEN DEMAND OF THE OHIO RIVER

The results of oxygen demand observations in the Ohio River and at the mouths of certain major tributaries, which are given in Table No. 1, columns 8, 9, and 10, are presented in terms of the 24-hour loss of oxygen on incubation at 20° C. (column 8), and also in terms of the total oxygen demand as calculated from the 24-hour figure by taking the value of the deoxygenation coefficient (K_1) as being 0.1 (see p. 7), whence the 24-hour demand represents 20.6 per cent of the total. It may be noted in this connection that this value of (K_1) defines a deoxygenation curve coinciding exactly with the "relative stability" curve given in Standard Methods of Water Analysis of the American Public Health Association (1920 edition, Table No. 15, p. 70). In column 10 of Table No. 1 the total oxygen demand figures of column 9, in terms of milligrams per liter, have been weighted in each case by the discharge of the river, in thousand second-feet, observed at the particular sampling station coincidently with the laboratory determination. The purpose of these weighted figures is to show the total amounts of unoxidized organic matter carried by the river at various points, corrected to a common basis of dilution; in other words, to eliminate changes in dilution as factors in masking real changes in the oxygen demand of the river between successive stations. Strictly speaking, the unit chosen for this purpose is one of rate; that is, it measures the quantity of biologically oxidizable matter (in terms of oxygen demand) carried by the river past a given point in each unit of time. Thus, the figures in column 10 may be reduced to terms of grams per second by multiplying them by the factor 28.3. For convenience, however, these values will be designated as "quantity units."

Corrected in this manner, an increase in the number of "quantity units" of oxygen demand as observed between two stations indicates that added amounts of oxidizable matter have been brought into the river at intermediate points either in the form of sewage and other wastes discharged directly into the stream, or through the medium

of tributaries and local surface drainage. A reduction, on the other hand, indicates that progressive satisfaction of oxygen demand has more than counterbalanced the effects of intermediate inflow in the river stretch in question. If the oxygen demand data be interpreted from this standpoint, some highly significant facts are disclosed as to the sources and behavior of oxidizable matter in the Ohio River.

In certain stretches of the river, relatively undisturbed by inflowing pollution, marked and consistent reductions in its oxygen demand are observable during periods of settled flow conditions. Among the shorter stretches, this tendency is particularly noticeable in those extending from stations 3 to 11, below Pittsburgh, and stations 475 to 488, below Cincinnati, where decreases in the oxygen demand of the river were observed consistently during the summer and autumn months of 1914.

Of the longer stretches, which are in general subject to relatively large additions of inflowing drainage water, even during low water, the only one in which a reduction in the number of "quantity units" of oxygen demand was observed with any degree of consistency was in the stretch, approximately 300 miles long, extending from Station 619, below Louisville, to Station 904, a comparatively short distance above the mouth of the Ohio River. During the period of June 1 to October 15, 1914, the average number of "quantity units" of oxygen demand observed at Station 619 was 81.3, or 2,300 grams per second. At Station 904 the average number during the same period was 50.6, or 1,432 grams per second. In spite of the large volumes of pollution brought into the Ohio between the two points by the sewage of Evansville, Henderson, and other cities, and by the Salt, Wabash, and numerous smaller rivers, the observed reduction in oxygen demand in this stretch of the river was about 38 per cent. and it would undoubtedly have been much greater in the absence of the intermediate sources of pollution noted.

In longer stretches of the river farther upstream, as, for example, between Station 104, below Wheeling, and Station 349, immediately above the Scioto River, the influx of a number of highly polluted tributaries, adding a large volume of flow to the Ohio, has a constant tendency to mask the true effects of organic oxidation going on in the stream, as is evidenced by the increase in the "quantity units" of oxygen demand between these two stations even during the summer low-water period of 1914. A similar tendency, though not so consistently marked, was observable in the long stretch of the river extending from Station 492, below the mouth of the Miami River, to Station 598, above Louisville.

In one stretch of the Ohio River receiving almost solely rural drainage, namely between Station 358, below the Scioto River, and Station 461, above Cincinnati, the increase in oxygen-demand values

observed during the study was so marked as to indicate that a very considerable proportion of the total oxidizable organic content of the river originates in surface drainage from sparsely inhabited and unsewered areas. This conclusion is borne out by the fact that the Ohio River carried, on the average, a very considerable excess of oxygen demand over that which would be accounted for by the total sewage contribution of the watershed. At Station 461, above Cincinnati, for example, the oxygen demand of the river during the year extending from May, 1914, to April, 1915, inclusive, averaged 352 grams daily per capita of sewered population in the entire watershed above this point. From observations immediately above and below Cincinnati during the low-water period of June 1 to October 15, 1914, the average amount of oxygen demand contributed to the river by the Cincinnati metropolitan district during this period was equivalent to 140 grams per capita daily, which, it may be noted, does not greatly exceed the figures for normal sewage, 112 grams and 100 grams per capita, given on pages 2–3. Taking these values as representing the normal contribution of oxygen demand in the form of sewage, the large excess of this constituent carried by the Ohio River, over that which would be accounted for in terms of sewage, is evident.

It may therefore be said that while evidences of the forces of progressive oxidation at work in the Ohio River are discernible in certain stretches of the stream and under conditions wherein stability of flow exists, these evidences are for the most part masked by disturbances resulting from inflowing pollution originating both in sewage and in surface drainage water.

Bearing in mind these factors, it is now proposed to examine the data with reference to the validity of an assumption made in deriving the formulas on pages 18–19; namely, that the rate at which oxygen demand is satisfied in the stream is governed by the same law that controls its rate of satisfaction under conditions of the laboratory. This law has been stated on page 5 to be as follows:

> The rate of biochemical oxidation of organic matter is proportional to the remaining concentration of unoxidized substance, measured in terms of oxidizability.

Or, in mathematical terms (p. 5):

$$\log \frac{L'}{L} = Kt$$

Owing to limitations above stated, the oxygen demand values observed at the various Ohio River stations must be sifted carefully for evidences of disturbance from surface inflow before being subjected to so severe a test as the one proposed. It is desirable, moreover, to have available for the purpose data covering a fairly long period, a

year at least. The only data from the Ohio River studies fulfilling these specifications are those which were obtained in the river stretch below Cincinnati extending from Station 475 to Station 488, a distance of, roughly, 13 miles. About midway between these two points was Station 482, but since it was located but a short distance below Government Dam No. 37, over which the water has a free fall during periods of low water, the oxygen demand results obtained at this station were to some extent influenced by the presence of small amounts of entrained air in the samples and were therefore not as reliable as those obtained at Stations 475 and 488. For these two stations results were available over the full period of May, 1914, to April, 1915, inclusive, as given in table No. 1, and in addition, for the months of May and June, 1915, during which period certain special observations were continued in the river below Cincinnati.

Considering these monthly average figures at Stations 475 and 488, it was obvious that if the rate at which the oxygen demand of the river was satisfied in this stretch were a logarithmic function of time, in accordance with the formula above stated, a close linear correlation should exist between the observed times of flow between the two stations, corresponding to (t) in the formula, and the logarithms of the ratio:

$$\frac{\text{Oxygen demand at Station 475}}{\text{Oxygen demand at Station 488}}$$

corresponding to $(\log \frac{L'}{L})$ in the formula. Employing the Galton-Pearson coefficient (r) as an index of this correlation, a value of $(r) = 0.85 \pm 0.043$ was obtained from the observed results. Since a value equal to unity would denote perfect correlation, it is apparent that the value obtained indicates a high degree of correlation between the two variables as observed in the river; so high, in fact, as to leave little doubt concerning the agreement between actuality and theory in this case.

A more severe test of this correlation was provided by an analysis of the same data from another viewpoint. Referring to Table No. 4, in which the results of this study are tabulated, the observed monthly reductions in oxygen demand between the two stations were converted to terms of percentage of the initial value (column 5) and reduced to a common basis of loss in 24 hours at 20° C. (column 8), using for this correction the formulæ and temperature factors described on pages 7–8. Referring to column 8 of the table, it is noted that during the period June–November, 1914, when flow conditions in the river were stabilized, the decrease averaged 23.5 per cent in 24 hours. At the mean temperature observed at Station 475 during this period (21.3° C.) the value of K_1, as given

in Figure No. 3, would be 0.116, and the 24-hour decrease, assuming a logarithmic rate, would theoretically be 22 per cent. For other months than the June–November period in 1914, a wide variation in the 24-hour figures was found, but it will be noted that during these months flow conditions in the river were unsettled, and observed times of flow between the two stations were so low that large errors of extrapolation would be involved in converting the results to a 24-hour basis.

TABLE No. 4.—*Reduction in oxygen demand between Ohio River Stations 475 and 488, as observed and as corrected to a uniform basis of time equivalent to 24 hours and temperature 20° C.*

[Arranged in order of magnitude in times of flow]

Month	Oxygen demand, parts per million observed		Decrease between stations		Time of flow, days, Stations 475 to 488		Per cent decrease, 24 hours, 20° C. (at observed rates)
	Station 475	Station 488	Parts per million	Per cent of initial	At river temperature	Equivalent at 20° C.	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
I. High river stages, shorter times of flow:							
February, 1915	7.76	7.08	0.68	8.8	0.18	0.11	72.5
January, 1915	10.30	8.92	1.38	13.4	.21	.09	134.8
May, 1914	3.84	3.11	.73	19.0	.23	.21	81.5
March, 1915	7.18	6.26	.92	11.4	.28	.14	71.1
December, 1914	12.14	10.78	1.36	11.2	.29	.15	65.6
June, 1915	3.45	2.86	.59	17.1	.31	.34	47.1
May, 1915	5.59	4.42	.17	30.4	.37	.35	80.4
April, 1915	5.72	4.76	.96	16.8	.43	.30	60.6
II. Low river stages, longer times of flow:							
June, 1914	4.37	3.60	.77	17.6	.73	.94	18.6
July, 1914	3.94	3.30	.64	16.3	.82	1.12	14.7
September, 1914	4.32	3.01	1.31	30.3	.88	.98	30.9
August, 1914	3.84	3.15	.69	18.3	.98	1.28	14.8
October, 1914	7.62	5.05	2.57	33.7	.99	.99	33.8
November, 1914	11.85	9.62	2.23	18.8	1.15	.70	25.8
Oct. 1-15, 1914	6.41	3.98	2.43	37.9	1.62	1.56	25.8
Mean, June-November, 1914							23.5

Further and even more striking evidence of the closeness with which the rate at which oxygen demand is satisfied in streams follows the logarithmic law has been afforded by a preliminary analysis of extensive data obtained by the United States Public Health Service from studies of the Illinois River during the years 1921 and 1922. Observations in a stretch of this river about 70 miles long, extending from La Salle to a point a short distance above Peoria, where conditions affecting natural purification are in a very fair state of equilibrium, were especially illuminating on this point. Taking the mean oxygen demand values observed at La Salle (Station 227) as the 100 per cent point, and computing the oxygen demand amounts observed at three other stations downstream as per-

centages of the La Salle figure, these results have been compared for the period of December, 1921, to April, 1922, with corresponding percentages calculated on the assumption that a logarithmic rate of decrease in the demand held throughout this river stretch. The results obtained are as follows:

Station	Time of flow	5-day oxygen demand, per cent of Station 227 value	
		Observed	Assuming logarithmic rate of decrease
	Hours		
227 (La Salle)	0	100	100
196 (Henry)	45	84	71
179 (Chillicothe)	103	43	47
166 (Averyville)	144	35	[1] 35

[1] Assumed the same as the observed figure in order to calculate values at Stations 196 and 179.

The closeness of agreement in the observed and calculated values at the two intermediate stations, 196 and 179, is the more striking in view of the relatively high river stages and low temperatures prevailing throughout the period of the observations. A similar comparison, based on observations during the summer of 1921 and 1922, was less satisfactory in view of an observed increase in oxygen demand between Stations 179 and 166 during this period, probably as a result of the intensive growth and decomposition of microscopic plants in this stretch of the river, which in the summer presents conditions approaching more closely those of a shallow lake than those of a flowing stream. Based on the summer observations between Stations 227, 196, and 179, however, the observed oxygen demand at the intermediate station, 196, was 85 per cent of the Station 227 figure, as compared with a calculated value of 81 per cent, assuming a logarithmic rate of decrease. Inasmuch as the total decrease in demand between Stations 227 and 179 amounted to 34 per cent of the Station 227 value, the close agreement of the calculated with the observed demand at Station 196 is significant.

In considering the foregoing data it should be borne in mind that observations of this kind made in natural streams are subject to inherent errors such as would cause inevitable variations in oxygen-demand results obtained even under stable conditions of flow. The use of such results for the purpose of verifying their conformance to a fixed mathematical law, as has been done in the present case, constitutes an extremely severe test, to be interpreted rather broadly and with due allowance for the inevitable "spread" of the data. Viewed in this light, the evidence presented

above affords ample justification for the assumption made in deriving formula (1).

THE OXYGEN RESOURCES OF THE OHIO RIVER

It has already been shown that the total oxygen resources of a stream at a given point are, first, the dissolved oxygen actually contained in the water at that point and, second, the potential resources available through the capacity of the stream for reaeration. The data relative to these two sources of oxygen will be separately discussed.

Dissolved oxygen.—In Table No. 1 the dissolved oxygen values at various stations throughout the Ohio and at various times during the period of the observations are shown in three ways, viz:

(*a*) In parts per million;

(*b*) In percentage of saturation; and

(*c*) As the saturation deficit in parts per million.

It is to be noted that the concentration of dissolved oxygen itself becomes critically low only during average and extreme low water and even then only in comparatively short stretches of the river below Pittsburgh and Cincinnati. The conditions for a few miles below Pittsburgh approached complete exhaustion of dissolved oxygen at extreme low water during October, 1914. Indeed, it is doubtful whether the present comparative freedom from general nuisance at this point would be enjoyed were it not for the bactericidal influence of the acid conditions in the Pittsburgh district. The lowest average oxygen value recorded at station 475, about 7 miles below Cincinnati, was 52 per cent of saturation, observed during the first half of October, 1914. As far as could be determined, no offensive conditions existed in the Ohio River below Cincinnati at that time.

During low river stages the waters at the mouths of several of the tributaries were found to be at a point of lower oxygen concentration than the waters of the main stream. This is noticeably true of the Cumberland and Tennessee Rivers, entering the Ohio near its mouth, at which point the main stream usually has recovered its dissolved oxygen to an extent of from 85 per cent to over 95 per cent of its saturation value.

A study of the saturation deficit figures is instructive in showing the rapid rate at which the Ohio recovers its oxygen supply through reaeration, even during low water periods. Thus, as indicated in Table No. 1, it appears that at station 461, above Cincinnati, the stream is often in better condition as regards reserve oxygen than it is above Wheeling, and is sometimes in better condition at station 904, near its mouth and above the Cumberland and Tennessee Rivers than above Cincinnati. These facts are of special significance in connection with the importance of the reaeration factor.

Reaeration.—A direct computation of the amount of reaeration in the river, expressed in any convenient concentration unit, may be readily made by applying the principle illustrated in Figure No. 5 and determining the difference between the actual oxygen content at the lower station and its calculated value in the absence of reaeration, as determined from the average oxygen demand for the given river stretch.

The actual amounts of reaeration per mile of river length observed during various months and in various stretches of the Ohio are given in Table No. 5. These amounts are rather widely dispersed, as would be expected in view of the marked variations with respect to oxygen saturation deficiency and to physical conditions of channel and flow encountered in the several river stretches and at different times. The following further analysis of the data was undertaken to determine if the reaeration values obtained are consistent with the known physical features of the various river stretches.

TABLE No. 5.—*Average rates of reaeration, in parts per million of dissolved oxygen, per mile of river length, observed in designated stretches of the Ohio River during specified months of 1914 and 1915.*

Stretch of Ohio River	Length (miles)	Average rate of reaeration, in parts per million of oxygen, per mile of river length												
		May (1914)	June	July	August	September	October 1–15	October 16–31	November	December	January (1915)	February	March	April
Stations:														
3 to 11	8.7	0.04	0.28	0.45	0.69	0.65	0.93	.....	.....	.....	.....	.....	.....	.....
11 to 19	7.3	.02	.07	.14	.30	.25	.20	.....	.....	.....	.....	.....	.....	.....
23 to 65	42.2	.002	.03	.03	.06	.04	.08	.....	.....	.....	.....	.....	.....	.....
65 to 77	11.9	.03	.09	.10	.07	.09	.16	.....	.....	.....	.....	.....	.....	.....
77 to 88	10.8	.03	.06	.06	.05	.09	.12	.....	.....	.....	.....	.....	.....	.....
104 to 349	245.7	.01	.07	.16	.51	.13	[1] 2.52	.....	.....	.....	.....	.....	.....	.....
349 to 461	112.0	.005	.02	.02	.02	.01	.03	.....	.....	.....	.....	.....	.....	.....
475 to 482	7.3	.01	.15	.20	.30	.23	.....	0.29	0.27	0.05	0.01	0.01	0.02	0.03
487 to 498	5.9	.007	.007	.03	.02	.01	.....	.04	.008	.01	.002	.006	.008	.01
493 to 598	105.4	.01	.13	.22	.26	.11	.....	.08	.18	.03	.01	.009	.03	.03
598 to 611	13.2	.001	.08	.11	.06	.07	.....	.18	.20	.03	.01	.07	.02	.07

[1] Probably in error by a wide margin owing to the excessively long indicated time of flow.

Because of the varying and complicating influence of these numerous factors, the reaeration values themselves are only of special local significance. A study of derived values of the reaeration coefficient, however, and of its relations to depth and turbulence, furnishes data of fundamental importance, and attention will therefore be directed to this phase of the subject.

The reaeration coefficient.—Values of the reaeration coefficient (K_2), calculated by the method outlined on pages 20–27, are given in Table No. 3. In Table No. 2 are also given the various factors employed in the calculation, transcribed from Table No. 1, and in

the last column, values of (K_2) corrected to a standard temperature, 20° C., by means of the curve in Fig. No. 7. The proper symbols for the various quantities in formula (1) are given at the heads of their respective columns.

On referring to Table No. 2, it will be noted that the assigned values of (L_a) for the two longer river stretches, stations 104–349 and 492–598, are so high in some cases as to appear rather incongruous when visualized in terms of actual oxygen concentration in the stream. This is notably true of the high value of (L_a) given for the river stretch, stations 104–349, for the period October 1–15, 1914. On referring to column 5 of the table it will be noted that coincident with these excessively high values of (L_a) are times of flow ranging from slightly less than 10 days to nearly 30 days, whereas the times of flow coinciding with lower values of (L_a), both in these two and the other river stretches, are, in general, of a lower order of magnitude.

In interpreting this apparent abnormality in the (L_a) figures, the significance of (L_a) as defined on page 18 and as further interpreted on pages 21–26 should be borne in mind. Being an hypothetical value, it represents, in all cases, an assumed quantity sufficient in magnitude, if concentrated immediately below Station A, to cause a residual oxygen demand at Station B equal to the arithmetical mean of (L_B) and (L'_A). Where the time factor is long, as in the two river stretches above cited, the quantity thus assumed is necessarily higher than would be the case if the time were shorter, and it may be much higher than the total oxygen demand actually introduced into the stream at points intermediate between A and B, causing the divergence of (L_B) from (L'_A). Perhaps the best evidence that these excessively high values of (L_a) are not abnormal is afforded by the general consistency of values of (K_2) derived from them, as compared with corresponding values derived for the same river stretch during months of shortened times of flow and lower values of (L_a).

Taking as an example the results of (K_2) calculations in the stations 104–349 stretch, as given in Table No. 2, column 8, comparison of the (K_2) value, 0.18, obtained for May, 1914, (L_a) being 3.8, with the value, 0.20, for October, (L_a) being 618.2, is significant evidence on this point. As previously noted, however (see footnote, p. 23), errors of observation and extrapolation are most likely to produce erroneous results when the time factor is long; hence the results of calculations in Table No. 2 for the two river stretches, stations 104–349 and 492–598, during the months of extreme low water in the summer of 1914 are to be regarded as being less reliable, on the whole, than corresponding results obtained for the shorter river stretches during the same period. These results have been retained for further analysis chiefly because of their general concordance

with the figures obtained for the shorter river stretches; likewise because they are desirable material for subsequent correlations to be described later in this text.

Inspection of the (K_2) values, corrected to a uniform temperature basis, shows that they are subject to considerable variation in each river stretch under the different flow conditions encountered from month to month. With a few exceptions, however, their general order of magnitude is about the same in different river stretches when determined under similar conditions of river stage, indicating that the conditions which cause variations in rates of reaeration are much more nearly uniform in different sections of the river under similar conditions of flow than they are in a given stretch at different river stages. In other words, those physical factors which change with river stage are the more influential upon reaeration rates.

The two stream conditions most affected by variations in gage height are depth and velocity of flow, the latter being closely related, of course, to turbulence. In Table No. 6 a fairly definite correlation is shown between values of (K_2) and both depths (H) and velocities of flow (V), the depth (H) being expressed as gage heights at river gages listed in the footnote of the table. The true nature of this relation, however, does not appear until each (K_2) value is weighted by the square of the depth (H^2), whereupon the quantity (K_2H^2) is found in every river stretch to approximate closely a simple power function of velocity (V), having the formula:

$$K_2H^2 = cV^n \qquad (4)$$

TABLE No. 6.—*Relation between the reaeration coefficient (K_2), the mean depth (H) of the stream above extreme low water, and the velocity of flow (V) in certain stretches of the Ohio River*

Notation:
K_2—Reaeration coefficient.
H—Mean depth in feet of stream above zero points of reference gages, which points are taken as representing extreme low water.
V—Mean velocity of flow of stream in feet per second.

Month	K_2 (20° C.)	H (feet)	K_2H^2	V (feet per second)
Station No. 3 to station No. 11:				
May, 1914	1.38	12.7	223.0	2.48
June	.35	16.4	94.0	.50
July	.35	16.3	66.3	.70
August	.17	16.3	45.1	.19
September	.18	16.6	49.6	.18
Oct. 1-15	.13	17.0	34.0	.09
Station No. 11 to station No. 19:				
May, 1914	.34	11.6	45.7	2.80
June	.12	4.8	2.80	.59
July	.20	3.6	2.60	.32
August	.14	3.1	1.35	.23
September	.14	2.6	.93	.19
Oct. 1-15	.06	2.5	.37	.10

TABLE No. 6.—*Relation between the reaeration coefficient (K_2), the mean depth (H) of the stream above extreme low water, and the velocity of flow (V) in certain stretches of the Ohio River*—Continued

Month	K_2 (20° C.)	H (feet)	KaH^n	V (feet per second)
Station No. 23 to station No. 65:				
May, 1914	.30	12.4	46.0	4.20
June	.33	6.0	9.0	1.36
July	.15	4.6	3.17	.88
August	.17	4.0	2.73	.58
September	.19	3.9	2.74	.43
Oct. 1–15	.48	1.4	.84	.20
Station No. 65 to station No. 77:				
May, 1914	1.11	11.4	144.0	3.64
June	.39	6.1	21.9	.71
July	.26	7.8	[illegible]6.9	.44
August	.17	8.2	1.4	.31
September	.33	6.9	18.7	.31
Oct. 1–15	.23	2.0	.91	.15
Station No. 77 to station No. 99:				
May, 1914	1.07	10.6	130.0	3.66
June	.71	5.6	22.30	1.01
July	.33	5.3	9.30	.62
August	.16	4.9	3.85	.40
September	.70	3.9	10.60	.40
Oct. 1–15	.16	3.4	1.85	.22
Station No. 104 to station No. 349:				
May, 1914	.24	14.3	49.0	3.72
June	.18	4.4	3.50	1.48
July	.12	3.8	1.73	1.18
August	.14	2.8	1.10	.91
September	.15	2.9	1.28	.96
Oct. 1–15	.32	1.1	.27	.51
Station No. 349 to station No. 461:				
May, 1914	.16	19.4	60.0	2.90
June	.65	5.4	19.0	2.36
July	.26	5.0	6.70	2.32
August	.15	4.0	2.40	1.96
September	.13	4.5	2.65	2.18
Oct. 1–15	.33	1.7	.95	1.41
Station No. 475 to station No. 482:				
May, 1914	.30	23.4	145.0	3.35
June	.22	10.1	32.6	.99
July	.33	10.9	39.2	.79
August	.26	10.5	28.7	.65
September	.40	10.8	48.6	.73
October	.73	10.6	82.0	.62
November	3.98	10.4	432.0	.54
December	.96	18.1	314.0	2.89
January, 1915	1.14	29.1	965.0	3.83
February	.61	34.8	738.0	4.20
March	1.43	18.0	484.0	3.74
April	1.60	11.5	144.0	1.73
Station No. 482 to station No. 489:				
May, 1914	.05	19.9	19.80	2.61
June		5.3		1.47
July	.13	4.9	3.12	1.40
August	.07	3.7	.98	1.23
September	.05	4.4	.97	1.34
October	.35	3.7	4.80	1.23
November	.61	2.8	6.25	1.12
December	.39	15.2	90.0	3.00
January, 1915	.10	28.2	68.50	4.11
February	.78	31.9	793.0	4.57
March	.71	18.4	168.0	3.00
April	.14	9.6	12.90	2.12
Station No. 492 to station No. 598:				
May, 1914	.27	19.9	107.0	3.26
June	.32	5.3	9.0	1.07
July	.37	4.9	13.7	.89
August	.46	3.7	6.3	.74
September	.61	4.4	11.8	.84
October	.31	[1] 5.8	10.3	.75
November		2.9		
December	1.05	15.2	242.0	2.48
January, 1915	.90	28.2	678.0	4.05
February	.59	31.9	600.0	4.51
March	.55	15.4	154.0	2.53
April	1.88	9.6	172.0	1.86

[1] Mean gage height at Dam No. 37 increased to allow for effects of local rains, as indicated by mean gage heights at the United States Weather Bureau gage, Madison, Ind.

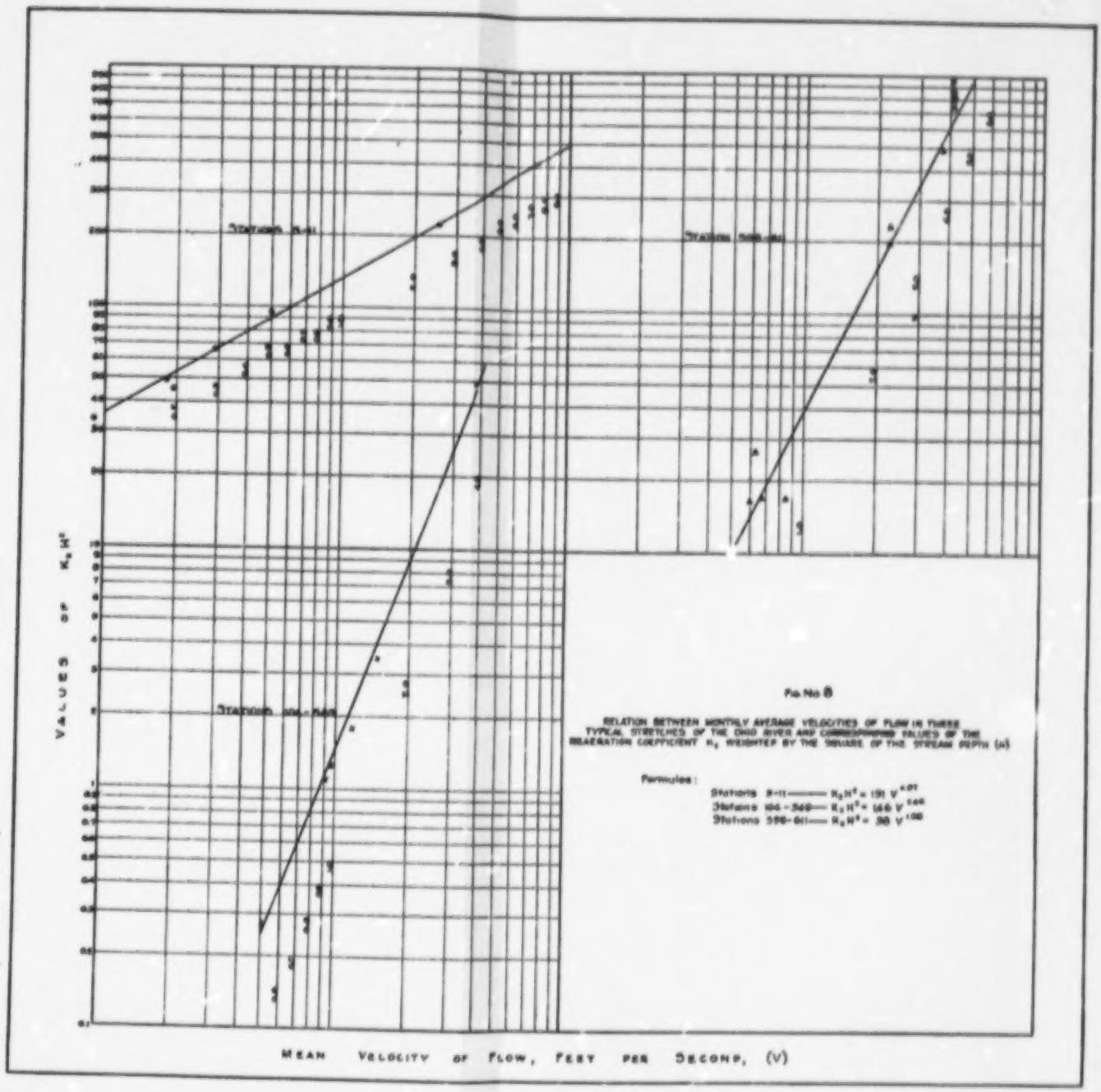

VALUES OF K_2H^3
MEAN VELOCITY OF FLOW, FEET PER SECOND, (V)
Fig No 8
RELATION BETWEEN MONTHLY AVERAGE VELOCITIES OF FLOW IN THREE TYPICAL STRETCHES OF THE OHIO RIVER AND CORRESPONDING VALUES OF THE REAERATION COEFFICIENT K_2 WEIGHTED BY THE SQUARE OF THE STREAM DEPTH (H)
Formulas:
Stations 3-11 —— $K_2H^3 = 191\ V^{1.01}$
Stations 184-368 —— $K_2H^3 = 166\ V^{1.46}$
Stations 598-611 —— $K_2H^3 = 38\ V^{1.00}$

TABLE No. 6.—*Relation between the reaeration coefficient (K_2), the mean depth (H) of the stream above extreme low water, and the velocity of flow (V) in certain stretches of the Ohio River*—Continued

Month	K_2 (20° C.)	H (feet)	$K_2 H^2$	V (feet per second)
Station No. 502 to station No. 611:				
May, 1914	.21	21.4	96.0	2.94
June	.36	6.9	17.1	.85
July	.47	6.0	17.0	.68
August	.43	5.1	11.2	.56
September	.87	5.5	26.3	.63
October	.54	5.5	16.4	.60
November		3.9		.39
December	.75	16.2	197.0	2.24
January, 1915	.58	29.1	490.0	3.73
February	2.44	36.1	3,180.0	4.11
March	.82	16.5	223.0	2.28

NOTE.—Value of (H) estimated as follows for the various river stretches:
Stations 3 to 11, from gage heights at Dam No. 3, upper gage.
Stations 11 to 19, from gage heights at Dam No. 4, upper gage, and Dam No. 6, lower gage.
Stations 23 to 65, from gage heights at Dam No. 8, lower gage.
Stations 65 to 77, from gage heights at United States Weather Bureau gage, Wheeling, W. Va.
Stations 77 to 88, from gage heights at Dam No. 17, middle gage.
Stations 104 to 349, from average of gage heights at reference gages from Dam No. 14 to Portsmouth, inclusive.
Stations 349 to 461, from average of gage heights at Portsmouth, Ohio, Maysville, Ky., and Dam No. 35, middle gage.
Stations 475 to 482, from weighted average of gage heights at United States Weather Bureau gage, Cincinnati, and at Dam No. 37, lower gage.
Stations 482 to 488, from gage heights at Dam No. 37, lower gage.
Stations 492 to 598, from gage heights at Dam No. 37, lower gage.
Stations 598 to 611, from gage heights at Dam No. 41, lower gage.

In Figure No. 8 are shown plots of (V) against (K_2H^2) for three stretches of the river selected at random, the data for these plots having been taken directly from the last two columns of Table No. 6. It will be noted that both ordinate and abscissa scales are logarithmic and that the plotted points follow closely a straight line, the equation of which is—

$$\log (K_2H^2) = n \log V + \log c.$$
$$\text{whence } K_2H^2 = cV^n \text{ as above}$$

If the term (H^2) be transposed to the denominator of the second member of the above equation, it will be noted that the value of (K_2) is inversely proportional to the square of the depth, as measured by the gage height (H). Bearing in mind that (K_2) is a direct measure of the rate of reaeration, this empirical finding is strikingly in accordance with the theory of the depth-square function as a factor in reaeration, stated on page 28, wherein it was shown that, viewed from standpoints of both diffusion and dilution, the reaeration rate should bear an inverse relation to a quantity approximating the square of the depth. This theory is further borne out by the fact that when the discharge of the Ohio River, corresponding to (Q), is substituted for the square of the gage height in formula (4), the quantity (K_2Q) bears the same kind of a power function relation to the observed velocity of flow as does (K_2H^2).

In Table No. 7 are given the empirical values of (c) and (n) for the different river stretches, as derived from plots of the data of

Table No. 6 similar to those shown in Figure No. 8. From the manner in which these coefficients have been obtained, wherein the biochemical factors controlling the oxygen demand and deficit values have been taken into account, it is apparent that (c) and (n) should be independent of such factors and should depend only upon the physical characteristics of the particular stream stretch for which they have been derived. The manner and extent of their dependence upon such characteristics will therefore be discussed.

TABLE No. 7.—*Values of (c) and (n) in relation:* $K_2=c\frac{V^n}{H^2}$, *for various stretches of the Ohio River*

Notation:
K_2 = reaeration coefficient.
V = mean velocity of flow, feet per second.
H = mean depth of water above extreme low water.

Stretch		Mean depth at extreme low water, (feet)[1]	(c)	(n)	Formula
From station—	To station—				
3	11	3.1	131.	0.57	$K_2=131\frac{V^{0.57}}{H^2}$
11	19	4.4	11.7	1.58	$K_2=11.7\frac{V^{1.58}}{H^2}$
23	65	5.2	6.2	1.39	$K_2=6.2\frac{V^{1.39}}{H^2}$
65	77	6.2	38.0	1.00	$K_2=38.0\frac{V^{1.00}}{H^2}$
77	88	7.4	18.0	1.48	$K_2=18.0\frac{V^{1.48}}{H^2}$
104	349	8.8	1.46	2.62	$K_2=1.46\frac{V^{2.62}}{H^2}$
349	461	9.5	0.23	4.06	$K_2=0.23\frac{V^{4.06}}{H^2}$
475	482	12.8	60.0	1.71	$K_2=60.0\frac{V^{1.71}}{H^2}$
482	488	15.3	0.23	5.40	$K_2=0.23\frac{V^{5.40}}{H^2}$
492	598	15.2	18.0	2.35	$K_2=18.0\frac{V^{2.35}}{H^2}$
598	611	11.7	38.0	2.12	$K_2=38.0\frac{V^{2.12}}{H^2}$

[1] Mean depth of extreme low water corresponds to mean depth at zero reading of reference gages.

Values of the constants (c) and (n) which, it will be noted, modify only the velocity of flow function, depend primarily upon physical stream conditions which influence turbulence. Broadly, the exponent (n) defines the range of variation in rates of reaeration in a given river stretch under different flow conditions, while the coefficient (c) defines their general order of magnitude. Physical conditions which influence turbulence in the same river stretch under various flow conditions should therefore modify (n) to the greater extent, while those which determine the effects of the same flow conditions upon turbulence in various river stretches should exert the more influence upon (c).

The effect of a given variation in river stage upon the velocity of flow is an important factor modifying turbulence in a given

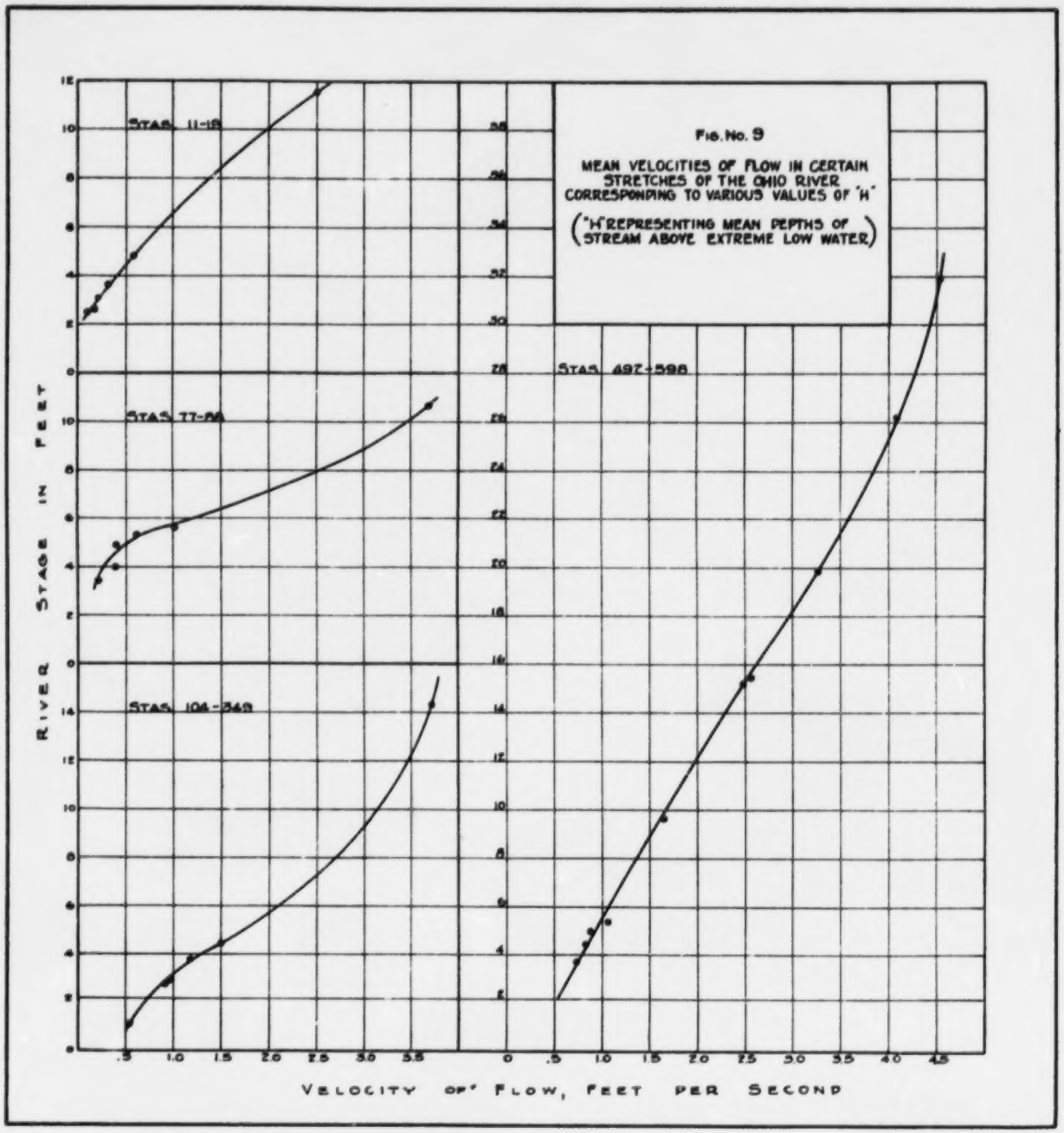


FIG. No. 9
MEAN VELOCITIES OF FLOW IN CERTAIN STRETCHES OF THE OHIO RIVER CORRESPONDING TO VARIOUS VALUES OF "H"
("H" REPRESENTING MEAN DEPTHS OF STREAM ABOVE EXTREME LOW WATER)
STAS. 11-18
STAS. 77-88
STAS. 104-348
STAS. 497-598
RIVER STAGE IN FEET
VELOCITY OF FLOW, FEET PER SECOND

stretch under various flow conditions, and this effect should therefore be closely related to the value of (n). The physical condition which influences the river stage-velocity relation is principally the shape of the channel. For example, channels with level bottoms and straight, steep side slopes produce an entirely different river stage-velocity relation than irregular channels with flat side slopes. In the latter instance it is common to observe conditions in which the cross-sectional area of flow of a rising river increases more rapidly than the discharge, with a resulting decrease in the actual mean velocity. In many instances in which the channel side slopes are straight and steep, as is frequently the case in the Ohio River, the velocity is closely proportional to river stage.

The river stage-velocity relations in the various Ohio River stretches for which values of (c) and (n) have been obtained are illustrated by the curves shown in Figure No. 9, which show the variations of this relation in different stretches of the Ohio. In Table No. 8 are summarized the relative increases in velocity produced by specified increases of gage height in the different river stretches. The mean value for each stretch is an index of the river stage-velocity relation, and their relation to corresponding (n) values, as shown in Table No. 9 and Figure No. 10, is definite and fairly consistent, considering the comparative roughness of the index. The data from the stations 3–11 stretch are omitted, its river stage-velocity relation being very poorly defined because of the manipulation of dams during the greater part of the period of study. The greater part of the stretch from stations 475 to 482 is also under the influence of backwater from Dam No. 37 during pool stages of the river. For this reason the data from this stretch have been separated into two groups, one of which applies to "dam down" conditions and the other to "dam up" conditions.

TABLE No. 8.—*Increase in mean velocity of flow, feet per second, in certain Ohio River stretches as related to increase in gage height (feet)*

Stations, Ohio River stretch	Increase in velocity of flow per 5-foot increase in gage height		
	Gage heights, 5 feet to 10 feet	Gage heights, 10 feet to 20 feet	Mean
11–19	3.1	.....	3.1
23–65	3.2	.....	3.2
65–77	6.4	.....	6.4
77–88	6.3	.....	6.3
104–349	1.8	.....	1.8
349–461	1.4	1.2	1.3
475–482 (dam up)	.....	7.2	7.2
475–482 (dam down)	.....	2.0	2.0
482–488	1.5	1.7	1.6
492–506	1.8	2.0	1.9
506–611	2.6	2.0	2.3

TABLE No. 9.—*Relation between mean increase*[1] *in velocity of flow for given increase of gage height in certain Ohio River stretches and corresponding values of* (n)[2] *in reaeration formula* ($K_2H^x=cV^n$), *tabulated according to magnitude of values of* (n)

Ohio river stretch		Value of (n)	Mean velocity increase with 5-foot increase of river stage
Serial No.	Stations		
8-a	[3] 475–482	[4] Negative	7.2
4	65– 77	1.00	6.4
3	23– 65	1.39	3.2
5	77– 88	1.48	6.3
2	11– 19	1.58	3.1
11	596–611	1.96	2.3
8-b	[5] 475–482	2.00	2.0
10	492–596	2.35	1.9
6	104–349	2.62	1.8
7	349–461	4.06	1.3
9	482–488	5.40	1.6

[1] Derived from data in Table No. 8.
[2] Derived from data in Table No. 7.
[3] Dam up.
[4] Probably due to influence of algæ growths in October and November. Allowing for this influence, value of (n) probably positive and very low.
[5] Dam down.

As regards the coefficient (c), it has already been noted that its value is defined largely by physical characteristics producing different degrees of turbulence in various river stretches under similar flow conditions. The slope and irregularity of the channel are probably the most prominent of such characteristics. The relation of these two factors to values of (c) in various stretches of the Ohio is therefore a matter of importance.

The numerical expression of the average slope of a river channel or of the water surface is simple, a convenient unit being in terms of feet vertical drop per mile of horizontal length. Measurement of the average slope of an irregular channel bottom is subject, however, to considerable error, since the elevation of the bottom at terminal points of a given stretch may not be representative, but may be determined by some local irregularity of profile rather than by the general slope of the channel. For this reason, the slope of the water surface at extreme low river stage is probably the more reliable index of the general slope of the bottom, allowing for whatever change occurs in the general depth of the water.

The irregularity of the channel, on the other hand, is not subject to the definiteness of expression that may be employed for the slope, since it is in itself an abstract, relative characteristic. Certain arbitrary methods of expression may be resorted to, however, which will permit comparison of the approximate irregularities of various stretches of river channel. Such methods are subject to much

variation, and the results derived through any one of them are to be regarded as merely rough indices of actual conditions. As regards its influence upon the turbulence of a stream, the unevenness of the channel is defined not only by the frequency of slope irregularities but also by their relative sizes and depths from the water surface. Obviously the expression of all of these factors, constantly varying in natural streams, is futile. A workable compromise which may be conveniently adopted consists of noting the number, per unit of

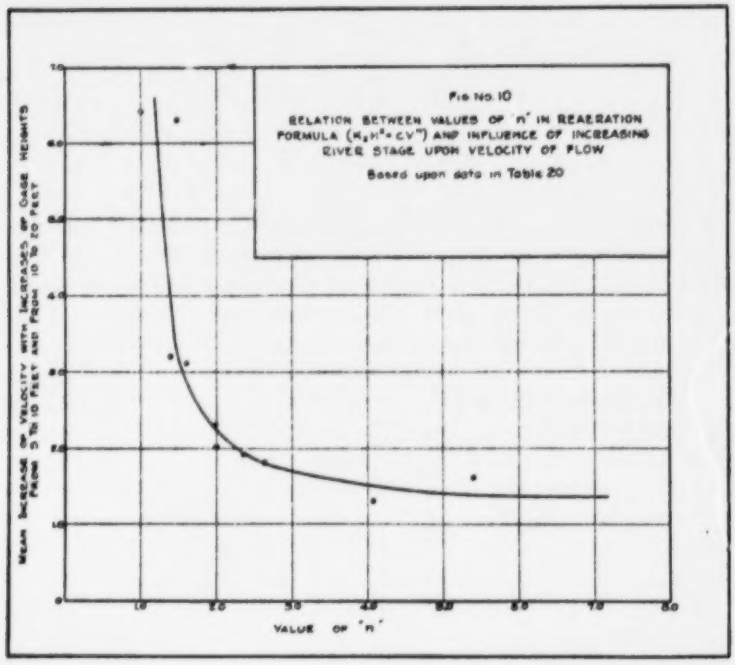


Fig No. 10
RELATION BETWEEN VALUES OF "n" IN REAERATION FORMULA ($K_2H^{1.5}=CV^n$) AND INFLUENCE OF INCREASING RIVER STAGE UPON VELOCITY OF FLOW
Based upon data in Table 20
MEAN INCREASE OF VELOCITY WITH INCREASES OF GAGE HEIGHTS FROM 5 TO 10 FEET AND FROM 10 TO 20 FEET
VALUE OF "n"

length, of slope irregularities, greater in size than a specified minimum, and assuming that in fairly large groups their average size and depth from the surface will be similar. For example, from a longitudinal profile of the channel bottom the number of changes in slope per mile may be counted, each one of which produces a change of elevation exceeding 1 foot. For convenience, this number may and will hereafter be designated as the "irregularity factor." In Table No. 10, values of this factor are given for the various stretches of the river enumerated in preceding tables.

TABLE No. 10.—*Values of "irregularity" factor for given Ohio River stretches, consisting of number per river mile of changes in longitudinal slope of channel bottom resulting in over one foot change in elevation*

Ohio river stretch		Length (miles)	Total	Changes in direction of channel bottom per mile (irregularity factor)
Serial No.	Station			
1	3–11	7.1	33	4.6
2	11–19	7.3	19	2.6
3	23–65	42.2	123	2.5
4	65–77	11.9	39	2.8
5	77–88	10.8	44	4.1
6	104–349	145.7	375	2.6
7	349–461	127.4	246	2.2
8	475–482	7.3	36	4.9
9	482–488	5.9	14	2.4
10	492–598	105.4	328	3.1
11	598–611	13.2	38	2.9

In Table No. 11 these same river stretches are divided into two groups according to values of the "irregularity factor" and arranged in each group in the order of decreasing low water slopes, the value of (c) being given also in each case. The first group, comprising stretches having factors between the limits 3 and 5, may be regarded as relatively uneven, while the second and larger group, with factors between 2 and 3, may be classified as relatively smooth. Both groups present, however, the fairly smooth type of channel found in the Ohio River and similar streams. The irregularity factor is taken account of in this manner, and attention may then be directed to a study of the relations between slope factors and values of (c) in each group. This relationship is shown in the table and is illustrated graphically in Figure No. 11, in which the characteristic curve for each group is drawn through the plotted values. Each relation is shown to be definite and the curves themselves are consistent with each other; for example, the curve of the higher irregularity group (3.0 to 5.0) gives higher values of (c) with equal slopes. The method of dividing the results into groups defined by limiting values of the irregularity factor is employed for the reason that the factors themselves are but approximate indices of channel unevenness and hence individual values are not in themselves highly significant. In streams of the type of the Ohio, lack of smoothness of the channel is probably not as important a factor in reaeration as is watercourses having rough, irregular channels. In some of the latter type, the frequency and sharpness of horizontal bends is also probably a factor modifying the influence of channel roughness upon the turbulence of the stream.

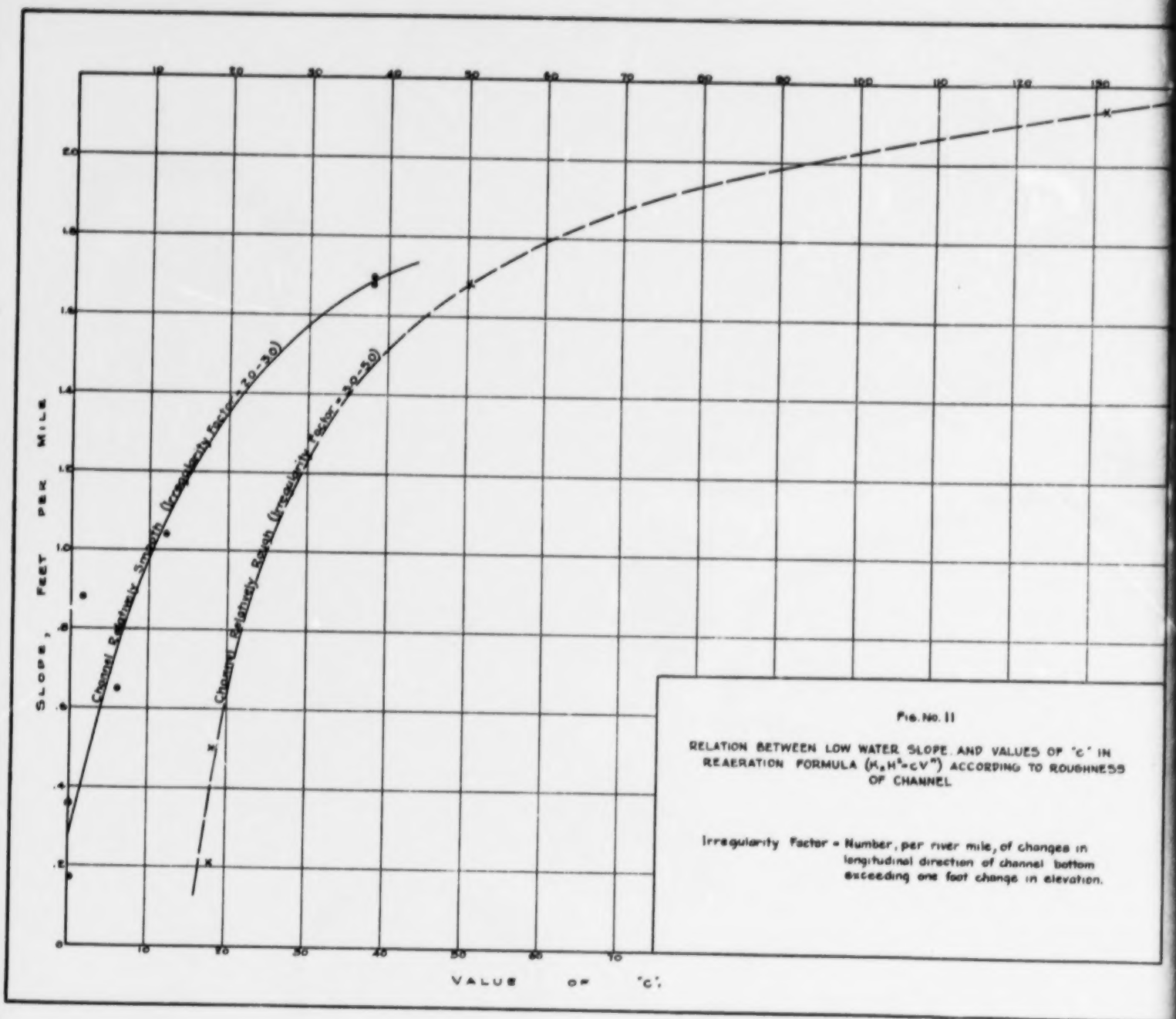

SLOPE, FEET PER MILE
VALUE OF 'C'
Channel Relatively Smooth (Irregularity Factor - 20-30)
Channel Relatively Rough (Irregularity Factor - 30-50)

Fig. No. 11

RELATION BETWEEN LOW WATER SLOPE AND VALUES OF 'C' IN REAERATION FORMULA ($K_2H^3=cV^n$) ACCORDING TO ROUGHNESS OF CHANNEL

Irregularity Factor = Number, per river mile, of changes in longitudinal direction of channel bottom exceeding one foot change in elevation.

TABLE No. 11.—*Relations between value of (c) in reaeration formula ($K_2H^3=cV^n$) and low-water slope in feet per mile, for two general types of stretches of Ohio River, grouped according to relative irregularities of channel*

Ohio River stretch		Value of (c)	Low-water slope (feet per mile)	"Irregularity" factor [1]
Serial No.	Stretch			
Relatively uneven "irregularity factors" [1] ranging from 3.0 to 5.0				
1	3- 11	131.0	2.14	4.6
8	[2] 475-482	50.0	1.60	4.9
5	77- 88	18.0	.50	4.1
10	492-508	18.0	.31	3.1
Relatively smooth, "irregularity factors" ranging from 2.0 to 3.0				
4	65- 77	38.0	1.08	2.8
11	598-611	38.0	1.67	2.9
2	11- 19	12.0	1.04	2.6
3	23- 65	6.0	.65	2.5
6	104-349	1.5	.88	2.6
7	349-461	.2	.36	2.2
9	482-488	.2	.17	2.4

[1] "Irregularity" factor = number, per river mile, of changes in longitudinal direction of channel bottom. (See Table No. 10.)
[2] See p. 54.
[3] For open-channel conditions (Dam 37 lowered).

From the relationship curves derived in Figures Nos. 10 and 11 it appears that whatever may be the weight of other factors in modifying the control exerted by stream depth and velocity of flow over rates of reaeration in the Ohio River (and there are doubtless other factors involved) the influences of channel slope and irregularity and of variations in river stage as related to changes in velocity are manifest and well defined. This is true, as will be shown later, to the extent that, given these three factors in the terms stated, values may be assigned to (c) and (n) in the formula:

$$K_2=\frac{cV^n}{H^3}$$

sufficiently close to give a very fair approximation of the true rate of reaeration of the Ohio under any given condition of flow. By deriving empirical formulæ for the curves above noted, this operation may be done more directly.

The relationship curve for values of (n) in Figure No. 10 is a hyperbola having the general form: $(x-a)(y-b)=1$. For this curve, $(a)=1$ and $(b)=1.17$, approximately. Taking the exponent (n) as (x) in the general equation, the following empirical formula is readily derived:

$$n=1+\frac{1}{y-1.17}$$

or,

$$n=\frac{y-0.17}{y-1.17}$$

in which (n) represents the abscissae and (y) the ordinates of the curve as shown in the chart.

The two (c) curves in Figure No. 11 are more difficult to formulate and unfortunately are not well fitted by empirical formulæ having exactly the same type of equation. Curve A, for smoother channels, has the more complex formula which is that of an exponential function with an additive constant, having the general equation: $y=a(10^{bx})+c$. For the curve in question, $a=0.39$, $b=1.16$, and $c=17$, approximately. Denoting (x) by the coefficient (c) and (y) by the slope (S), we then have:

$$c=0.39(10^{1.16})+17 \tag{5}$$

The formula of curve B is less complicated, being a simple power function of the general form: $y=ax^m$. For this curve, $a=1.1$ and $m=2.3$, whence (using the same notation as above):

$$c=1.1S^{2.3} \tag{6}$$

Having derived formulas (4), (5), and (6), the equivalents of (c) and (n) may be substituted in the equation:

$$K_2=\frac{cV^3}{H^3}$$

giving the following empirical formulas for (K_2) directly in terms of its determining factors:

1. For relatively smooth channel (irregularity factor=2.0 to 3.0),

$$K_2=\frac{[0.39(10^{1.16}S)+17]V^{\frac{y-0.17}{y-1.17}}}{H^3} \tag{7}$$

2. For relatively irregular channel (irregularity factor=3.1 to 5.0),

$$K_2=\frac{[1.1S^{2.3}]V^{\frac{y-0.17}{y-1.17}}}{H^3} \tag{8}$$

in which—

K_2=reaeration coefficient.

V =velocity of flow in feet per second.

H =mean river depth, in feet, above extreme low water (ordinarily given by means readings of gages in or near the river stretch considered).

S=low-water slope of the channel, in feet per mile.

y=mean relative increase in velocity of flow per 5 feet increase in gage height.

The formulæ (7) and (8) have been derived principally in order to provide a concise mathematical statement of the empirical relations modifying the reaeration coefficient (K_2). While probably less accurate for purposes of computation than are the curves them-

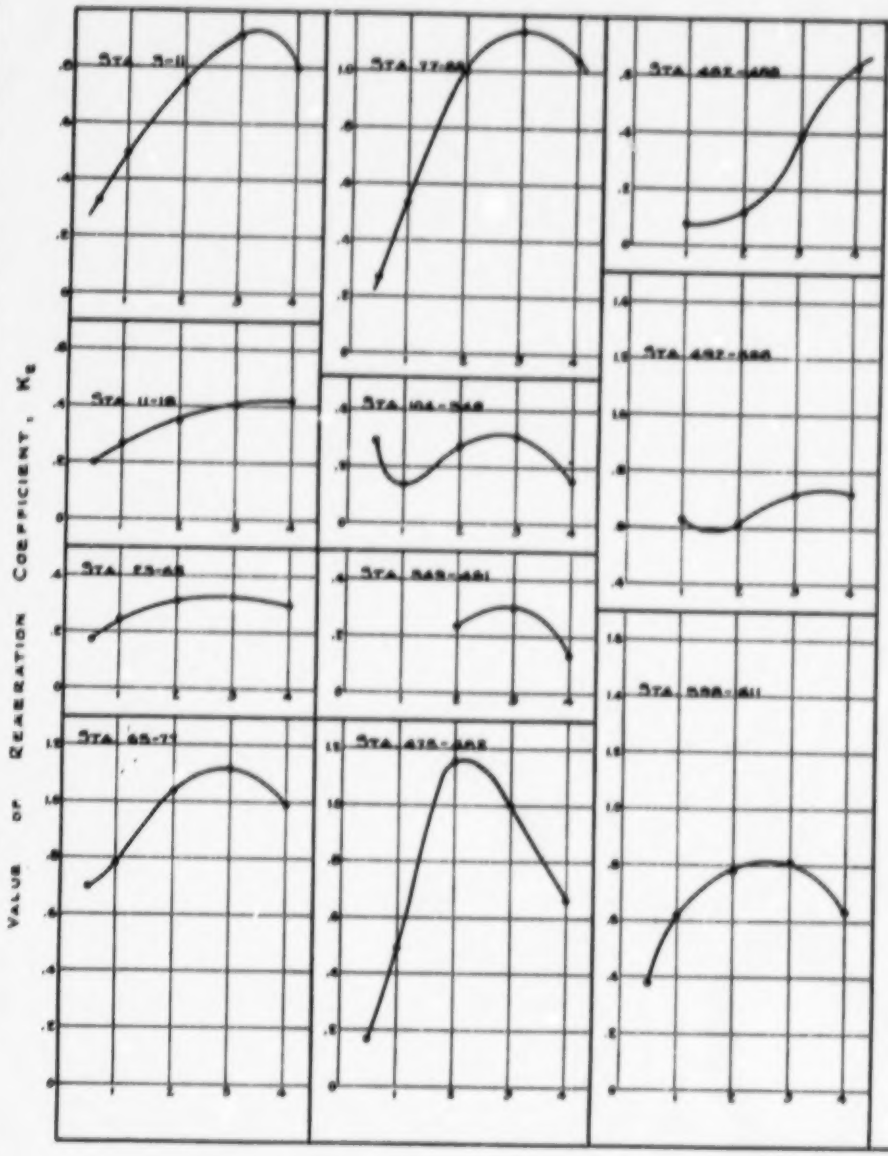


Fig. No. 12.

VALUES OF REAERATION COEFFICIENT, K_2, IN CERTAIN STRETCHES OF OHIO RIVER CORRESPONDING TO VARIOUS ACTUAL MEAN VELOCITIES OF FLOW IN FEET PER SECOND

Based upon data in Table 23

Note. Values of K_2 shown in this chart are derived from application of general equation $K_2H^n = cV^n$ with values of 'c' and 'n' for each stretch as given in Table No. and values of 'H' derived from curves in Fig. No. Each value of 'H' represents the mean depth of the stream above extreme low water

selves, they are sufficiently well correlated with the observed data for all practical purposes of estimate. In applying the formulas to the Ohio River it is evidently preferable, where possible, to choose for a given river stretch the specific values of (c) and (n) which have been derived for it. Since, as shown in Figure No. 9, the gage height (H) and the velocity (V) are definitely related to each other, a given reading of the former in a particular section of the river fixes the value of the latter, permitting the direct calculation of the reaeration coefficient (K_2) for a given river-stage condition by means of the formula:

$$K_2 = \frac{cV^n}{H^3}$$

Employing this general method, except that depths (H) were derived from assumed velocities of flow (V), values of the reaeration coefficient (K_2) were computed for a number of stretches of the Ohio River under different velocity-of-flow conditions, with results as given in Table No. 12 and shown graphically in Figure No. 12. In Table No. 13 are tabulated values of (H) in the several river stretches corresponding to the different velocities of flow assumed in obtaining Table No. 12, these values having been derived from the relation curves of Figure No. 9. The purpose of drawing the curves of Figure No. 12 has been to show the relations existing between variations in flow conditions, as indicated by velocities of flow, and induced rates of reaeration, as measured by values of the reaeration coefficient (K_2).

TABLE No. 12.—*Calculated values of reaeration coefficient in specified stretches of the Ohio River under different velocity-of-flow conditions, assuming stream-depth factors H corresponding to velocities as defined by curves of Figure No. 9*

Serial No.	Ohio River stretch	Value of K_2 at velocities (feet per second)				
		0.5	1.0	2.0	3.0	4.0
	Stations:					
1	3 to 11	0.33	0.50	0.74	0.92	0.90
2	11 to 19	.20	.27	.35	.40	.42
3	29 to 65	.17	.25	.31	.32	.30
4	65 to 77	.70	.78	1.05	1.13	.99
5	77 to 98	.27	.54	1.00	1.14	1.04
6	104 to 349	.20	.14	.28	.31	.15
7	349 to 461			.24	.28	.13
8	475 to 482	.17	.49	1.16	.99	.66
9	482 to 488		.07	.12	.38	.63
10	492 to 598	1.36	.63	.62	.72	.72
11	598 to 611	.36	.62	.78	.81	.64

TABLE No. 13.—*Mean depths (H) of Ohio River above extreme low water in certain stretches corresponding to given velocities of flow*

[Derived from Figure No. 9]

Serial No.	Ohio River stretch	Value of H at velocities				
		0.5	1.0	2.0	3.0	4.0
	Stations:					
1	3 to 11	[1] 16.3	[1] 16.3	[1] 16.3	[1] 16.3	19.0
2	11 to 19	4.4	6.6	10.0	12.8	15.8
3	23 to 65	3.9	5.0	7.2	9.4	12.0
4	65 to 77	5.2	7.0	8.5	10.1	12.4
5	77 to 88	4.9	6.8	7.1	9.0	11.6
6	104 to 349	.9	3.2	5.7	9.2	19.2
7	349 to 461	.....	.....	4.0	8.4	22.2
8	475 to 482	[1] 10.3	[1] 11.1	13.0	19.9	31.1
9	482 to 492	.....	1.8	8.8	15.2	24.8
10	492 to 598	1.6	5.4	12.1	18.2	28.5
11	598 to 611	4.8	7.8	14.6	22.0	33.4

[1] Pool stage. Dam up.

One characteristic of these curves deserving comment is the tendency of (K_2) to reach a maximum with increasing velocities of flow up to a certain point, beyond which lower values are obtained. The velocity of flow coinciding with this maximum (K_2) point varies in different river stretches studied, but in a majority of them it is reached at somewhere between 2 and 4 feet per second. It thus appears that at a certain critical velocity, peculiar to a given stream type, the induced rate of reaeration reaches its maximum, but that when the velocity is greater or less than this critical value, reaeration proceeds more slowly.

The reasons for this apparently paradoxical situation are to be found in the relation existing in a given river stretch between the two terms (cV^n) and (H^2), in the second member of the reaeration formula:

$$K_2=\frac{cV^n}{H^2}$$

In this formula it will be noted that if (V) and (H) increase at the same rate, the value of (K_2) increases or diminishes according to whether the exponent of (V), which is (n), is greater or less than the exponent of (H), which is 2.0. If the exponent of (V), however, is less than 2.0, (K_2) increases if (V) increases at a sufficiently greater rate than (H). The converse is likewise true; that is, (K_2) may actually decrease in value with an exponent of (V) greater than 2.0 if (H) increases at a sufficiently greater rate than (V).

Under natural flow conditions it is found (see fig. No. 12) that in a few stretches, as between Stations 23 and 65, the velocity (V) increases at about the same rate as the depth (H) above a certain low depth, but in a majority of the stretches (V) increases in relation to (H) at greatly decreasing rates above a certain point; hence

with increasing river stages the effect of depth upon (K_2) becomes increasingly manifest above this point, while that of velocity becomes less so. In other words, at low depths the velocity effect predominates, but at greater depths the depth factor predominates. The velocity-depth relations themselves are influenced almost entirely by the shape of the channel cross section, as has already been pointed out.

Another characteristic of the reaeration-velocity curves is found in their reaching of a definite minimum point at some low velocity. Apparently below this point the effect of decreasing depth predominates over that of diminishing velocity. However, these extremely low portions of the curves are probably not as reliable as the portions above this minimum point; in fact, it is likely that at the minimum point the effect of velocity becomes almost negligible and reaeration tends to follow the laws of diffusion in quiescent waters.

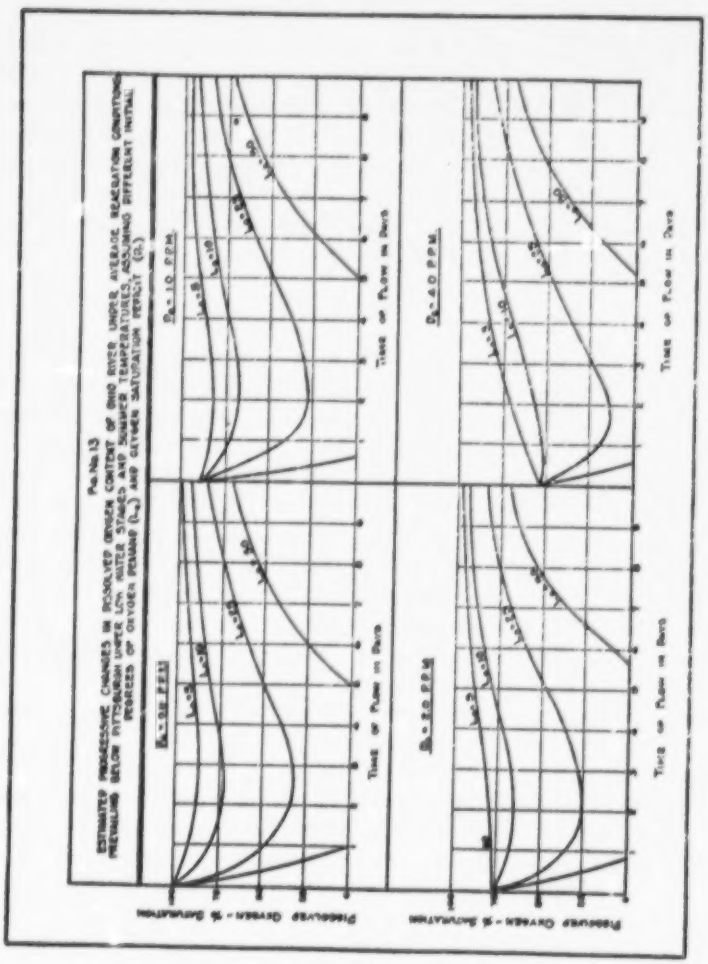
APPLICATIONS OF DATA

It has been shown in the foregoing text that the oxygen self-purification of the Ohio River is a measurable phenomenon, governed by definite laws and proceeding according to certain fundamental physical and biochemical reactions. Because of the fundamental character of these reactions and laws, it is fairly evident that the principles underlying the phenomenon as a whole are applicable to virtually all polluted streams. In considering the applications of the data which may be made, it is therefore important to recognize the relation which they may bear not only to the problems of the Ohio River but also to those of other streams in general.

As regards the Ohio River, no widespread and detailed applications of the data were justified under conditions of pollution of this stream, as existing at the time of these observations, for the reason that it was not, as a whole, seriously overburdened from a standpoint of its oxygen reserve. This is amply shown by the dissolved oxygen results which have been cited. In only two short sections of the river, below Pittsburgh and Cincinnati, respectively, did conditions at any season of the year approach serious oxygen depletion. With the possible exception of Pittsburgh, where the situation is complicated by a chemical withdrawal of oxygen from the river as a step in the oxidation and precipitation of soluble iron salts, conditions nowhere approached an acute stage. For these reasons any applications which may be made of the data must deal largely with hypothetical cases. In this connection, however, it is instructive to show, by applying the principles developed from the study, approximately what dissolved oxygen conditions may be expected in the river stretches below the two cities under various conditions of

pollution and to compare them in this respect under similar pollution conditions.

The most critical seasonal condition with respect to deoxygenation of the river consists of dry-weather flow at summer temperatures. Referring to Table No. 3, the average reaeration capacities of the

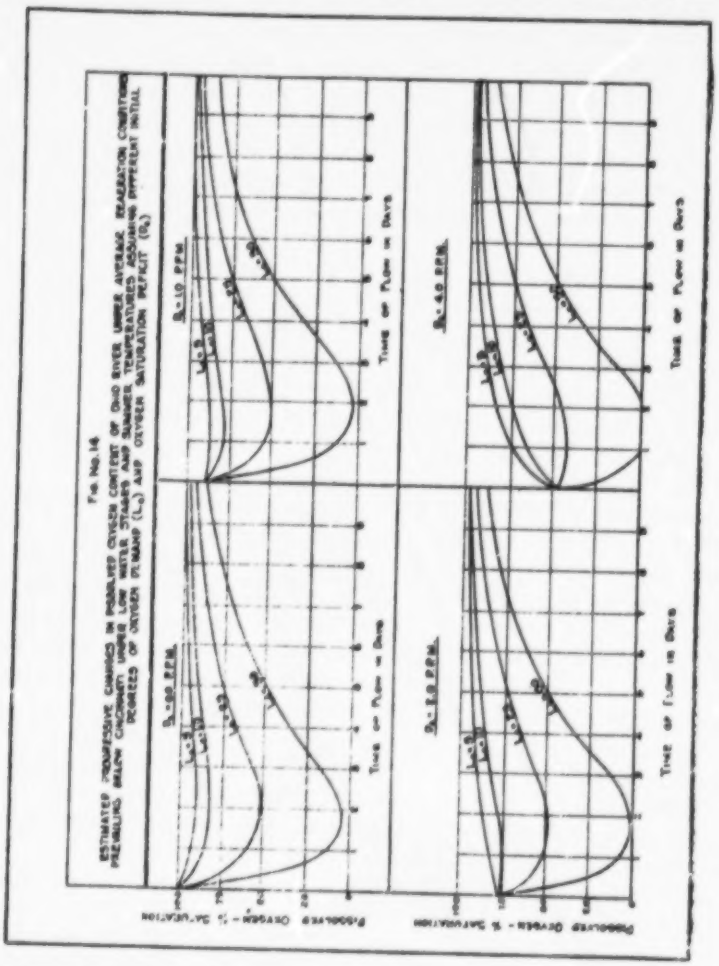


river below Pittsburgh and Cincinnati, respectively, during the July–October period in 1914, are shown by the following mean values of the reaeration coefficient, K_2:

Below Pittsburgh, $K_2=0.25$ (approximately).
Below Cincinnati, $K_2=0.45$ (approximately).

At the mean temperatures prevailing in the river during this period the rate of deoxygenation below both cities is given by a value of the coefficient K_1, equal to 0.12.

Referring to the general formula (1) on page 18, and assuming the above constants, values of the oxygen saturation deficit (D) after



various times of flow (t) may be readily computed for different assumed values of initial oxygen deficit (D_a) and initial oxygen demand (L_a). Assuming these values and computing D, results are obtained as shown graphically in Figures Nos. 13 and 14, the former chart illustrating conditions below Pittsburgh and the latter parallel conditions below Cincinnati.

In constructing the two charts it has been assumed that all of the pollution enters the river at a single point and is immediately mixed thoroughly with the stream water, producing the initial oxygen demand and oxygen-saturation values employed. This assumption is, of course, not in accordance with the actual circumstances, since the wastes of both cities are discharged into the river along frontages of several miles. In order to obtain proper values (L_a) and (D_a), it would be necessary to make a detailed study of the amount and composition of the wastes discharged at each sewer outfall and also of the times of flow of the river between the various outfalls.

In spite of the crudeness of the assumptions upon which they are based, the two charts bring out several interesting points concerning the reaeration capacity of the river as related to its permissible pollution. Thus, it is indicated that no dangerously low depletion of the dissolved oxygen supply of the river below the two cities may be expected until the amounts of pollution added by them are sufficient to produce an initial oxygen demand in the river of 25 parts per million or thereabouts. Assuming the oxygen demand of normal city sewage to be 1.2 grams per capita daily, this would represent a critical dilution of 11 parts of river water to 1 part of sewage. The present low-water dilution at both cities is much higher than this figure.

Under actual circumstances it is, of course, certain that locally offensive conditions may be expected to prevail in the river at both cities long before a general exhaustion of its reserve oxygen supply is threatened at points below them. With sluggish conditions of flow the rate of mixture of sewage with the river is extremely slow, and more or less localized zones are formed around the sewer outlets within which sludge deposition and deoxygenation of the supernatant water proceeds far more actively than at points in the stream farther removed. Such zones are very likely to become foci of local offense, while the stream as a whole still maintains a considerable amount of reserve oxygen. For this reason a detailed study of local conditions is essential to the fixing of safe minimum standards of residual oxygen supply below any given city.

As regards the application of the various constants to calculations of oxygen changes in different stretches of the Ohio River, it is of interest to note the approximate precision obtainable therefrom in so far as it is indicated by the deviation of calculated from observed oxygen values at the lower terminals of those river stretches for which data were obtained in connection with the present study. In order to make such a comparison values of the velocity-gage height factor (y) and the slope (S) in formulas (4) and (5), respectively, were taken for each river stretch from Tables Nos. 9 and 11, and

from them values of (c) and (n) were derived, using the curves of Figures Nos. 10 and 11. From the (c) and (n) values thus obtained reaeration coefficients (K_2) at the stream temperature were computed for several river stretches and various months, following the detailed method outlined in Appendix C. Employing these calculated (K_2) values, the oxygen content of the river at the lower terminal of a particular stretch was computed from the observed oxygen deficit at the upper station, employing formula (1) for this purpose. The calculated figure was then compared with the corresponding dissolved oxygen content as actually observed at the lower station, with results as indicated in Table No. 14.

TABLE No. 14.—*Comparison of the observed dissolved oxygen content of the Ohio River at various sampling stations with corresponding values calculated by the use of formulae (1), (7), and (8), basing the calculations in each case on the observed oxygen content at the next station upstream*

Station	Month	Dissolved oxygen per cent saturation		Deviation of calculated from observed			
		Observed	Calculated	Per cent saturation		Per cent of observed (Δ_2)	(Δ^2)
				Plus	Minus		
19	September, 1914	88.0	77.1		10.9	−12.4	153.76
508	May, 1914	83.4	93.3	9.9		+11.9	141.61
65	September, 1914	91.6	82.8		8.8	−9.6	92.16
508	April, 1915	97.6	90.7		6.9	−7.1	50.41
508	December, 1914	89.5	83.3		6.2	−6.9	47.61
509	June, 1914	90.2	96.2	6.0		+6.6	43.56
65	August, 1914	91.1	85.5		5.6	−6.1	37.21
65	October 1–15, 1914	94.7	90.0		4.7	−5.0	25.00
83	....do	88.5	93.0	3.5		+3.9	15.21
482	May, 1914	84.0	87.5	3.5		+4.2	17.64
508	October, 1914	91.5	88.0		3.5	−3.8	14.44
19	June, 1914	88.5	91.9	2.9		+3.4	11.56
88	August, 1914	91.5	94.1	2.6		+2.8	7.84
19	....do	81.1	78.8		2.3	−2.8	7.84
88	September, 1914	96.3	94.2		2.1	−2.2	4.84
19	July, 1914	87.7	85.7		2.0	−2.3	5.29
65	....do	88.9	90.6	1.7		+1.9	3.61
508	January, 1915	91.4	89.7		1.7	−1.9	3.61
65	June, 1914	89.4	87.9		1.5	−1.7	2.89
508	February, 1915	91.9	90.5		1.4	−1.5	2.25
508	August, 1914	95.1	96.4	1.3		+1.4	1.96
482	December, 1914	81.1	82.3	1.2		+1.5	2.25
19	October 1–15, 1914	72.5	71.4		1.1	−1.5	2.25
88	May, 1914	90.9	90.2		.7	−.8	.64
88	June, 1914	95.5	94.8		.7	−.7	.49
88	July, 1914	93.8	94.4	.6		+.6	.36
482	April, 1914	94.6	95.0	.4		+.4	.16
65	May, 1914	87.7	88.0	.3		+.3	.09
482	February, 1914	95.0	95.2	.2		+.2	.04
508	July, 1914	95.4	95.2		.2	−.2	.04
508	September, 1914	97.0	96.8		.2	−.2	.04
19	May, 1914	92.8	92.7		.1	−.1	.01
482	March, 1915	95.1	95.2	.1		+.1	.01
508	....do	89.0	89.1	.1		+.1	.01
482	January, 1915	94.4	94.4			.0	.00
	Total number			15	19		
	Σx^2						606.69

Standard deviation $= \sqrt{\frac{606.69}{35}} = 4.5$ per cent.

Probable error $= 0.67449 \times 4.5 = 3.0$ per cent.

In order to show the distribution and range in magnitude of deviations of the calculated from observed values, the results in this table have been arranged in descending order of magnitude of the squared percentage deviations. It will be noted that the positive and negative deviations are fairly well balanced, considering the relatively small number of items. If these deviations be taken as a measure of the error of calculation, and if each one of the two groups, positive and negative, be classified according to the frequency of errors greater than varying amounts, a curve is obtained which coincides closely with the so-called "normal" curve of error. This being the case, it is permissible to determine the probable error by the conventional method, which consists of squaring the individual errors (v) and computing the standard deviation and probable error by the following relations:

$$\text{Standard deviation} = \sqrt{\frac{\Sigma v^2}{n}}$$

$$\text{Probable error} = 0.67449\sqrt{\frac{\Sigma v^2}{n}}$$

where (Σv^2) = sum of squares of deviations and (n) = number of items.

Applying this method, the standard deviation and probable error of the calculated oxygen values in Table No. 14 are found to be 4.5 and 3 per cent, respectively, as referred to the observed results. Taking into account the errors involved in the laboratory and physical determinations of the various factors in the formula, this composite error is remarkably low. The result obtained tends strongly to confirm the theory underlying the formulas applied, and to indicate that the errors involved in their application are, to a large extent, compensating ones. It should be pointed out, however, that a calculation such as the foregoing one is by no means a test of the general applicability of the formulæ and constants derived from this study, for reasons that are obvious. It constitutes, on the other hand, an excellent check upon the formulæ, as far as their application to the Ohio River is concerned.

As regards the more general application of the Ohio River constants to other streams, it is of interest to note the results of a test of this kind made in connection with an analysis of results obtained from a sanitary survey of the Thames River basin, in Connecticut, conducted during the summer of 1915. Among the data from this survey were gathered certain basic figures relative to the amounts of domestic sewage and idustrial wastes discharged into the Thames at various important points, the summer discharge of the stream and its tributaries, and the slope and general characteristics of its channel. From these data, values of the reaeration coefficient (K_2)

were derived and, proceeding from point to point in the stream, its theoretical dissolved oxygen content was computed, allowing for the inflowing of tributaries and the introduction of fresh pollution at the proper intervals. The oxygen values thus calculated were then compared with those actually observed by the State Board of Health, with results as given in Table No. 15.

TABLE No. 15.—*Comparison of calculated with observed dissolved oxygen results at specified points in Thames River Basin, Conn., calculated values being based on application of Formula (1)*

Station	Dissolved oxygen (parts per million)	
	Calculated [1]	Observed [2]
Above Webster	3.9	.....
French River at State line	.3	1.7
French River at mouth	1.6	2.8
Quinebaug River at State line	3.9	3.9
Quinebaug River above French River	5.4	4.8
Quinebaug River above Putnam	3.4	3.1
Quinebaug River above Five Mile River	5.4	3.4
Quinebaug River above Jewett City	5.5	3.2
Quinebaug River at mouth	5.5	3.6
Willimantic River at mouth	6.7	4.7
Shetucket River above Quinebaug River	7.4	4.7
Shetucket River below Quinebaug River	6.8	5.9
Mean	4.8	3.8

[1] Based on values: K_1=0.125.
K_2=0.50 and 0.96, for main and smaller streams, respectively.
[2] Observations by State board of health.

In interpreting these results it should be pointed out that a number of the factors entering into the calculations were virtually indeterminate, and broad assumptions in respect to them were necessary in order to complete the numerical computations. While there is considerable deviation between individual computed and observed values, there is unquestionably a marked correlation between them, as is evidenced by the fact that their coefficient of correlation [1] is 0.81 and its ratio to its probable error is 11.6. It will also be noted that if a certain fixed standard of minimum permissible dissolved oxygen content, say 3.0 or 4.0 parts per million, were to be employed as a basis for judgment regarding the condition of the stream at the several stations, the calculated results would lead to about the same conclusions as would the observed values, as regards the zones in which the oxygen status of the river were satisfactory or otherwise.

In general, the application to other streams of the constants derived for the Ohio River would require that particular care be observed in the selection of the different factors concerned with stream and channel characteristics, which may be expected to vary

[1] The Galton Pearson coefficient. (See p. 41.)

rather widely from those prevailing in the Ohio. In many cases, as in the example above given, certain of these characteristics can not be properly determined in definitely measurable terms without detailed local study. In this phase of the matter merely a beginning has been made in the Ohio River studies, and further work dealing with a large variety of stream types will be necessary before the entire working range of reaeration coefficients can be established.

SUMMARY AND CONCLUSIONS

The studies described in this paper have had as their chief aim the formulation of a reasonably complete set of working principles from which further studies of questions involving the organic stability of polluted streams may be made with more assurance than has hitherto been possible. In working out such a statement it has been necessary, in certain parts of the discussion, to reverse the usual procedure of developing a law from empirical findings, to the extent of evolving what may appear to be a rather elaborate theory in order to lay the groundwork for interpreting the findings themselves. This has been especially true of the portion of our discussion dealing with reaeration, wherein the establishment of a basis for numerically measuring the rate factor has necessitated the formulation of a reaeration coefficient from wholly theoretical considerations. The basic theory underlying practically the whole discussion, which is summarized in formula (1), is in reality, however, an extremely simple one when resolved into its component parts, and it is our belief that the experimental evidence on which it rests is fundamentally sound.

The empirical relationships established with respect to reaeration rates in the Ohio River must be interpreted, for the present at least, as applicable only to this stream or to rivers of a closely similar type as regards their physiography. Further studies of these relations will doubtless reveal wide variations in the specific values of constants derived for streams of different types, and will probably also disclose that formulas expressing them do not always follow the same mathematical laws as found in the present case. These matters can be cleared up only by additional observations on streams of widely different characteristics and by the checking of formulas against results of observations.

Certain basic units of measurement employed in connection with this study, especially those measures which apply to factors correlated with stream turbulence, need far more satisfactory definition than has been practicable in the present case. This statement applies in particular to the rather crude indices which have been employed for expressing channel irregularity and the velocity-depth relation. For the former, the use of the Kutter (n) or the Chezy

(c), and for the latter, the constants of the velocity-depth relation curve, might be advantageous from a standpoint of their more general applicability. Here again are questions for further study.

As regards future studies of the subject, that which is probably most needed at present is a further experimental investigation of a number of fundamental points bearing on both the biochemical oxidation and the physical reaeration of streams, employing improved laboratory methods which have been developed since the data for the present study were collected. Studies of this kind should comprehend the following main subjects:

(1) Coefficients of oxidation (corresponding to K_1) for various types of city sewage and industrial wastes.

(2) The temperature relations of both oxidation and reaeration coefficients.

(3) Rates of reaeration of bodies of water subjected to various controlled degrees of turbulence.

(4) The influence of various physical factors, operating in natural streams, upon reaeration rates, such as, for example, the action of dams, bends, channel obstructions of different kinds, and irregularities in the channel bottom. The modifying influence of variations in stream depth upon these phenomena should also be studied under controlled conditions.

Studies of this kind can be made to the best advantage by a combination of facilities afforded by the laboratory, the experimental tank and flume, and a small polluted natural stream, so situated as to offer opportunities for a close measurement and, if desired, a certain degree of artificial control, of the factors influencing natural purification phenomena. Further studies along these lines are contemplated in connection with the work of the stream pollution laboratory of the United States Public Health Service at Cincinnati, Ohio, where exceptionally good facilities are available for making them.

In the present text the applications of the various formulæ and constants to specific problems have been merely touched upon, attention in this respect having been confined to one or two examples, designed to illustrate methods and to indicate roughly the expected precision of calculations of the kind. Further studies of this character need to be made for a number of actual cases where the results of calculation may be checked against those of observation. From the few studies that have been made, it would appear that a high degree of precision is obtainable in estimating the conditions of pollution of the Ohio and its main tributaries under which any specified minimum reserve of dissolved oxygen may be maintained in this stream. It is also our belief that the application of the principles formulated in this paper to many stream pollution problems

of a localized nature involving the prevention of nuisance or protection of fish life, would greatly clarify the complex points at issue and assist in their practical solution. With this phase of the question our studies have thus far been but little concerned.

In conclusion, it may well be emphasized that the shortcomings of our analysis of the subject with which this paper deals are fully realized. Its results are presented, however, with the hope that, inadequate as they may be, they may assist toward a more rational treatment of the vexing problems of nuisance prevention and preservation of aquatic life which the alarmingly rapid increase in the pollution of many important rivers of the country has entailed. These problems are largely economic ones, and in solving them, full account should be taken of the enormous asset represented by the reaeration capacities of flowing streams. This asset is not only tangible but definitely measurable, and it represents a saving of millions of dollars to the country at large if properly utilized and rightly distributed. As regards the most advantageous plan of its utilization, and more especially of its just distribution, an engineering and legal problem is thereby presented which may well challenge the best thought among those charged with the public welfare.

APPENDIX A

SOLUTION OF DIFFERENTIAL EQUATION GIVING RATE OF CHANGE IN DISSOLVED OXYGEN WITH COMBINED REAERATION AND BIOCHEMICAL OXIDATION

Notation:

D_a = initial oxygen saturation deficit in parts per million.
D = oxygen saturation deficit at time (t) in parts per million.
K_1 = deoxygenation coefficient.
K_2 = reaeration coefficient.
t = time in days.
L_a = initial oxygen demand in parts per million.

Differential equation:

$$\frac{dD}{dt} = K_1L - K_2D$$

$$= K_1L_ae^{-K_1t} - K_2D \qquad (1)$$

This equation is linear, first order (Leibnitz), of general form—

$$\frac{dy}{dx} + Py = Q$$

(see Murray's "Differential Equations," pp. 26 and 27) where $y = D$, $x = t$, $P = K_2$, and $Q = K_1L_ae^{-K_1t}$.

Let $Q = 0$

Then $\frac{dD}{dt} + K_2D = 0$

$$\frac{dD}{dt} = -K_2D$$

$$\frac{dD}{D} = -K_2dt$$

$$\text{Log } D = -\int K_2dt$$

$$D = Ce^{-\int K_2dt} = Ce^{-K_2t}$$

$$De^{\int K_2dt} = C \qquad (2)$$

Differentiating (2)

$$e^{\int K_2dt}(dD + K_2Ddt) = 0$$

$$e^{\int K_2dt}(dD + K_2Ddt) = e^{\int K_2dt}K_1L_ae^{-K_1t}\,dt$$

$$De^{\int K_2dt} = \int e^{\int K_2dt}K_1L_ae^{-K_1t}\,dt + C$$

$$D = e^{-\int K_2dt}\left(\int e^{\int K_2dt}K_1L_ae^{-K_1t}\,dt + C\right)$$

$$= e^{-K_2t}\left(K_1L_a\int e^{K_2t}e^{-K_1t}\,dt + C\right)$$

$$= e^{-K_2t}\left(K_1L_a\int e^{(K_2-K_1)t}\,dt + C\right)$$

$$= e^{-K_2t}\left(\frac{K_1L_a}{K_2-K_1}e^{(K_2-K_1)t}\right) + Ce^{-K_2t} \qquad (3)$$

In order to check (3) at this point, differentiate it thus:

$$\frac{dD}{dt} = -\frac{K_1^2L_a}{K_2-K_1}e^{-K_1t} - CK_2e^{-K_2t} \qquad (4)$$

Multiplying (3) by (K_2).

$$K_2D = \frac{K_1K_2L_a}{K_2-K_1}e^{-K_1t} + CK_2e^{-K_2t} \qquad (5)$$

Adding (4) and (5):

$$\frac{dD}{dt} + K_2D = \frac{K_1(K_2-K_1)}{K_2-K_1}L_ae^{-K_1t}$$

$$= K_1L_ae^{-K_1t} \quad \text{(this is same as (1)).}$$

To determine the value of (C) from (3), let "t" = 0

Then $$D_a = \frac{K_1L_a}{K_2-K_1} + C$$

and $$C = D_a - \frac{K_1L_a}{K_2-K_1}$$

Then (3) becomes

$$D = \frac{K_1 L_a}{K_2 - K_1} e^{-K_1 t} - \frac{K_1 L_a}{K_2 - K_1} e^{-K_2 t} + D_a e^{-K_2 t} \quad (6)$$

$$= \frac{K_1 L_a}{K_2 - K_1} (e^{-K_1 t} - e^{-K_2 t}) + D_a e^{-K_2 t} \quad (7)$$

Appendix B

Reformulation of resultant oxygen formula (1)[1] for a condition such that any deviation of (L_B) from ($L_A e^{-K_1 t}$) is assumed to be due to pollution or dilution uniformly distributed along the stream between stations A and B. Comparison of results obtained by application of the equation thus developed, with results obtained by using formula (1).[1]

Given:

L_A = the observed oxygen demand at station A.

L_B = the observed oxygen demand at station B.

L'_A = the oxygen demand at station B as calculated from the observed value of (L_A) at station A.

t = time of flow, in days.

K_1 = coefficient of deoxygenation.

K_2 = coefficient of reaeration.

L_t = increase or decrease in oxygen demand produced up to time (t) by uniformly distributed increments of inflow.

p = increase or decrease in oxygen demand of stream, per unit of time, producing an effect equivalent to (L_t) at time (t).

At station B, $(L_t) = (L_B - L'_A)$.

Required:

Integrated form of differential equation $\frac{dD}{dt} = K_1 L - K_2 D$ for the condition assumed,

where D = oxygen saturation deficit at time (t)

and L = remaining oxygen demand at time (t).

For the assumed condition,

$$L = L_A e^{-K_1 t} + L_t$$

$$\text{or } \frac{dD}{dt} = K_1 (L_A e^{-K_1 t} + L_t) - K_2 D \quad (1)$$

The problem first becomes one of determining the value of (L_t) in terms of determinable factors. In order to derive an expression for (L_t) in terms of (p), the method of increments may be employed, assuming (L_t) to remain constant through each unit increment of time (Δt).

Then, where $t = 1$, $L_t = p$

where $t = 2$, $L_t = p + pe^{-K_1}$

where $t = 3$, $L_t = p + pe^{-K_1} + pe^{-2K_1}$

where $t = n$, $L_t = p + pe^{-K_1} + pe^{-2K_1} \ldots\ldots pe^{-(n-1)K_1}$

From this series the sum of the terms up to any time (t) may be derived thus:

$$L_t = \frac{p(1 - e^{-K_1 t})}{1 - e^{-K_1}} \quad (2)$$

at Station B, $\frac{p(1 - e^{-K_1 t})}{1 - e^{-K_1}} = L_B - L'_A$

$$\text{or, } p = \frac{(L_B - L'_A)(1 - e^{-K_1})}{1 - e^{-K_1 t}} \quad (3)$$

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[1] See p. 18.

Substituting the value of (L_t) from (2) and (1),

$$\frac{dD}{dt} = K_1\left[L_A e^{-x_1 t} + \frac{P(1-e^{-x_1 t})}{1-e^{-x_1}}\right] - K_2 D \qquad (4)$$

This question is a linear one of the first order, corresponding to Leibnitz's:

$$\frac{dy}{dx} + Py = Q \text{ where } x = t$$
$$y = D$$
$$P = K_2$$
$$Q = L_A e^{-x_1 t} + \frac{P(1-e^{-x_1 t})}{1-e^{-x_1}}$$

$$Let \left[L_A e^{-x_1 t} + \frac{P(1-e^{-x_1 t})}{1-e^{-x_1}}\right] = 0$$

$$Then \frac{dD}{dt} = -K_2 D$$

$$and \; De^{\int x_2 dt} = c \qquad (5)$$

Differentiating (5), we have

$$e^{\int x_2 dt}(dD + K_2 D dt) = 0$$
$$= e^{\int x_2 dt}.\, K_1\left(L_A e^{-x_1 t} + \frac{P(1-e^{-x_1 t})}{1-e^{-x_1}}\right)dt$$

$$De^{\int x_2 dt} = \int e^{\int x_2 dt}.\, K_1\left(L_A e^{-x_1 t} + \frac{P(1-e^{-x_1 t})}{1-e^{-x_1}}\right)dt + C'$$

$$D = e^{-\int x_2 dt}\int e^{\int x_2 dt}.\, K_1\left(L_A e^{-x_1 t} + \frac{P(1-e^{-x_1 t})}{1-e^{-x_1}}\right)dt + C' e^{\int x_2 dt}$$

$$= e^{-x_2 t}\left[\frac{K_1 L_A(1-e^{-x_1})-K_1 P}{(K_2-K_1)(1-e^{-x_1})}e^{(x_2-x_1)t} + \frac{K_1 P}{K_2(1-e^{-x_1})}.\; e^{x_2 t}\right] + C' e^{-x_2 t} \qquad (6)$$

To determine the value of (C'), let (t) = 0 in (6). Then (6) becomes

$$D = \frac{K_1 L_A(1-e^{-x_1}) - K_1 P}{(K_2-K_1)(1-e^{-x_1})}(e^{-x_1 t} - e^{-x_2 t}) + \frac{K_1 P}{K_2(1-e^{-x_1})}(1-e^{-x_2 t}) + D_A e^{-x_2 t} \qquad (7)$$

Now from (3),

$$P = \frac{(L_B - L'_A)(1-e^{-x_1 t})}{1-e^{-x_1 t}}$$

Substituting the equivalent of (p) into (7) and canceling out the terms $(1-e^{-x_1})$, we have at Station B,

$$D = \frac{K_1 L_A(1-e^{-x_2 t}) - K_1(L_B - L'_A)}{(K_2-K_1)(1-e^{-x_2 t})}(e^{-x_1 t} - e^{-x_2 t}) + \frac{K_1(L_B - L'_A)}{K_2(1-e^{-x_2 t})}(1-e^{-x_2 t}) + D_A e^{-x_2 t} \qquad (8)$$

$$= \frac{K_1 L_A}{K_2 - K_1}(e^{-x_1 t} - e^{-x_2 t}) + D_A e^{-x_2 t} + \left[\frac{K_1(L_B - L'_A)}{K_2(1-e^{-x_2 t})}(1-e^{-x_2 t}) - \frac{K_1(L_B - L'_A)}{(K_2-K_1)(1-e^{-x_2 t})}(e^{-x_1 t} - e^{-x_2 t})\right] \qquad (9)$$

Referring to the original formula (1), page 18, it is noted that equation (9) differs from it only in respect to the rather complex expression contained in the brackets, [ ], which expression represents the effect of the assumption of uniform distribution of inflow between stations "A" and ' ω.

Simplifying further equation (9) by substituting for (L'_A) its equivalent $(L_A e^{-x_1 t})$, we have

$$D = \frac{K_1 L_A}{K_2 - K_1}(e^{-x_1 t} - e^{-x_2 t}) + D_A e^{-x_2 t} + \left[\frac{K_1(L_B - L_A e^{-x_1 t})}{K_2(1-e^{-x_2 t})}(1-e^{-x_2 t}) - \frac{K_1(L_B - L_A e^{-x_1 t})}{(K_2-K_1)(1-e^{-x_2 t})}(e^{-x_1 t} - e^{-x_2 t})\right] \qquad (10)$$

Equation (10) reduces to the symmetrical form:

$$D = \frac{K_1[K_2(1-e^{-Kd})(L_B - L_A e^{-Kd}) - K_1(L_B - L_A e^{-Kd})(1-e^{-Kd})]}{K_2(K_2-K_1)(1-e^{-Kd})} + D_A e^{-Kd} \quad (11)$$

Comparison of results obtained in calculations of (K_2) for four stretches of the Ohio River, by use of formula (1), as applied in the main text and by use of equation (11), as developed by assuming a uniform distribution of inflow between stations A and B:

In order to test the application of the original resultant oxygen formula (1), employed in connection with the main text against results obtained by assuming that any deviation of the observed oxygen demand (L_B) at Station B from the Station A residual (L'_A) at Station B is due to a uniformly distributed inflow between the two points, a comparison was made[1] of (K_2) values obtained in four stretches of the Ohio River, first by using the original formula, (1) and, second, by employing equation (11), developed in this appendix.

To make the test severe, the longest two river stretches of the entire series (viz, stations 104–349 and stations 492–508) were included in the four stretches selected for comparative study. The other two stretches selected (stations 475–482 and stations 77–88) are of less than average length; hence the two extremes of stretch length are fairly represented. The data for the calculations were taken from Table No. 2, presented in connection with the main text. The method of calculation employed was similar to that which is outlined in Appendix C, modified in applying it to equation (11) only in respect to the terms used. (See derivation of equation (11), this appendix.)

A summary of values of (K_2) obtained by use of the two formulæ is given in the table following.

River stretch	Month	Value of formula (1) (a)	(K_2) by equation (11) (b)	Deviation (a) from (b)	Per cent deviation referred to (b)
Stations 104–349 (245 miles)	1914				
	May	0.180	0.165	+0.015	9.1
	June	.268	.299	−.031	10.4
	July	.302	.288	+.014	7.4
	August	.212	.190	+.022	11.6
	September	.173	.153	+.020	13.2
	October	.200	.187	+.013	6.9
Mean					9.8
Stations 492–508 (100 miles)	1914				
	June	.732	.709	−.1.6	34.8
	July	1.690	1.797	−.792	30.2
	August	.788	.515	+.273	53.0
	September	.751	1.084	−.333	30.8
	October	.247	.276	−.029	10.5
Mean					32.4
Stations 475–482 (7 miles)	1914				
	May	.725	.306	+.019	9.2
	June	.570	.573	−.005	.9
	July	.650	.657	−.007	1.1
	August	.585	.590	−.005	.9
	September	.514	.512	+.002	.4
	October	.601	.600	+.001	.1
	1915				
	May	.902	.906	−.008	.4
	June	1.005	1.056	−.051	4.8
Mean					2.3

[1] The calculations incidental to this comparison were made by Sanitary Engineer K. R. Crohurst, of the Public Health Service.

River stretch	Month	Value of formula (1) (a)	(K_2) by equation (11) (b)	Deviation (a) from (b)	Per cent deviation referred to (b)
	1914				
Stations 77-68 (11 miles)	July	.520	.541	−.012	3.2
	August	.245	.236	+.009	3.8
	September	.718	.704	−.006	2.7
	October	.144	.130	+.008	5.9
Mean					3.7

In interpreting the results above tabulated, it should be pointed out that, for purposes of application, values of (K_2) are not to be considered as being significant beyond the second decimal place; hence deviations in the third decimal place are insignificant. It will be noted further that the deviations obtained in the present series of comparisons, when taken as a whole, are fairly well balanced as regards algebraic sign, 11 of them being positive and 13 negative. This would appear to indicate that the differences in (K_2) values as obtained by the two methods are not due to any inherent source of error in one procedure or the other, but are more likely due to departures, in individual cases, of conditions of inflow and other disturbing factors from those which may be assumed in applying a particular formula. That the errors involved in either procedure are not serious as far as the final result is concerned is indicated by the close agreement of (K_2) values obtained by the two formulas in three out of the four cases for which the comparison was made. In one of the three cases where the agreement was reasonably close (via, the river stretch, stations 104-349) the degree of concordance between the two sets of figures is surprisingly high in view of the extreme length of this river stretch and the correspondingly long time period over which the calculation must be made.

In the single instance in which the agreement between the calculated values of (K_2) was not close (via, the stations 492-598 stretch) there is good reason for believing that the use of equation (11) would give results subject to a considerable margin of error, since a large proportion of the inflow entering the river stretch in question is discharged into it at a single point, about midway between the two terminal stations (through the mouth of the Kentucky River). An assumption of uniformly distributed inflow in a case of this kind, such as would be involved in the use of equation (11), would be an obvious source of error, tending to give higher than the true figure. On referring to the table it will be noted that in four of the five months for which the comparison was made for the stations 492-598 stretch the values of (K_2) obtained by use of equation (11) were higher than corresponding values obtained by formula (1).

With the one exception above noted, the differences in values of (K_2) as obtained by the two formulae are not strikingly great, their general agreement being well within the expected limits of precision in calculations of this kind. Furthermore, the evidence at hand does not indicate the existence of any fundamental source of unbalanced error in the use of formula (1), such as would vitiate it for purposes of calculation where conditions of disturbance in a given river stretch are indeterminate. For such cases, as well as the one most frequently encountered in practice, namely, a condition of massive pollution of a stream at certain well-defined points, the wisdom of employing formula (1), rather than the much more complex formulae of the type of equation (11), appears to be borne out rather strikingly by the results obtained from the foregoing comparison.

Appendix C

Method of derivation of reaeration coefficients for Ohio River

A. *Data.*—(1) Monthly mean initial dissolved oxygen content at upper and lower ends of stretch.

(2) Monthly mean final dissolved oxygen content at upper and lower ends of stretch, after 24 hours' incubation in closed bottle at 20° C. (Other incubation periods=2 or 5 days.)

(3) Monthly mean 24-hour loss of dissolved oxygen upon incubation at 20° C., obtained as differences of monthly mean initial and final dissolved oxygen content. (Other incubation periods=2 or 5 days.)

(4) Temperature of water at each end of stretch (monthly mean).

(5) Time interval of flow for stretch, in days (monthly mean).

(6) Mean velocity of flow for stretch in feet per second (monthly mean).

B. *Method of calculation.*—(1) Using value of $\theta=1.047$, correct 24-hour losses upon incubation from 20° C. to T° C. (river temperature), assuming $K_1=0.1$ for 20° C.

(K_1) is value in formula: $\text{Log}\,\frac{L'}{L}=K_1 t$

L'=initial total oxygen demand.

L =final total oxygen demand after time t.

(2) Referring to Figure No. 6 b, compute the total oxygen demand (L) from the observed 1, 2, or 5 day loss of dissolved oxygen on incubation, at 20° C., of samples collected at Station A. If the incubation temperature is 20° C.

$$L_A=\frac{\text{1-day loss}}{0.206}=\frac{\text{2-day loss}}{0.369}=\frac{\text{5-day loss}}{0.684}$$

(3) Compute the total oxygen demand (L_B) at Station B by a similar procedure.

(4) With the value of (K_1) at the river temperature, compute the value of (L_a) thus:

$$L_a=\frac{L_A+L_B(10K_1{}^t)}{2}$$

(5) Formula for (K_2):

$$D_b=\frac{L_a K_1}{K_2-K_1}(10^{-K_1 t}-10^{-K_2 t})+D_a\cdot 10^{-K_2 t}$$

D_a=dissolved oxygen saturation deficit at Station A (parts per million).

D_b=dissolved oxygen saturation deficit at Station B (parts per million).

L_a=mean total oxygen demand (parts per million) at Station (A), based on mean of (L_A) and (L_B).

K_1=deoxygenation coefficient at river temperature.

t =time from (A) to (B), in days.

K_2=reaeration coefficient at river temperature (unknown).

(2) Assume three or four values of (K_2) and solve for (D_b) in each case.

(3) Plot values of (K_2) and (D_b) obtained as above, and connect points with smooth curve.

(4) Take from the curve the value of (K_2) corresponding to the actual value of (D_b).

(5) Substitute the value of (K_2) obtained as in (4) in the formula and compute (D_b). If (K_2) is correct, (D_b) should be the same as the observed value.

Correction of (K_2) values for temperature: Use temperature relation curve, as given in Figure 7, correcting (K_2) from the river temperature to 20° C.

Correlation of (K_2) at 20° C. with mean velocities (V) and depths (H):

(1) General formula: $K_2 = \frac{cV^n}{H^x}$

where (V) = mean velocity in feet per second and (H) = mean depth of the river above extreme low water, taken in the present case as the gauge height at a convenient reference gauge for the river stretch in question.

(2) Plot corresponding values of (K_2H^x) and (V) on logarithmic paper, with (K_2) as ordinates and (V) as abscissae.

(3) Scale the slope of a line drawn through the plotted points. This slope, expressed as a ratio, equals the term (n) above.

(4) Determine (c), which is represented by the intercept on the vertical axis of the plot.

(5) The equation of (K_2) in terms of (V) and (H) is now defined, and the method of correlating (c) and (n) with physical conditions has been given in the main text.

C. *Numerical example*—

Station A = Station 3, Ohio River
Station B = Station 11, Ohio River } Month of July, 1914 (Table No. 1).

Mean temperature (T) = 24.6° C.
Time of flow (t) = 1.81 days.
(K_1) at 24.6° C. = 0.124.
$D_a = 8.36 - 4.36 = 4$ parts per million.
$D_b = 8.53 - 7.21 = 1.32$ parts per million.
$L_a = 2.73$ parts per million.

$$\frac{L_a}{K_2 - K_1} = \frac{0.335}{K_2 - 0.124} = a$$

$$10^{-K_2 t} = 0.598$$

Try $K_2 = 0.500$ $10^{-K_2 t} = 0.598$

$K_1 = 0.124$ $10^{-K_1 t} = 0.124$
$K_2 - K_1 = 0.376$ diff. $= 0.474$ $a = \frac{0.335}{0.376} = 0.89$

$D_a = 0.89 \times 0.474 + 4.00 \times 0.124 =$
$= 0.422 \times 0.496 =$
$= 0.918$

This value too low. Therefore (K_2) assumed too high.

Try $K_2 = 0.300$ $10^{-K_2 t} = 0.598$

$K_1 = 0.124$ $10^{-K_1 t} = 0.286$
$K_2 - K_1 = 0.176$ diff. $= 0.312$ $a = \frac{0.335}{0.176} = 1.90$

$D_a = 1.90 \times 0.312 \times$
$\times 4.00 \times 0.286 =$
$= 0.594 + 1.144 =$
$= 1.738$

This value still too low.

Try $K_2 = 0.200$ $10^{-K_2 t} = 0.598$

$K_1 = 0.124$ $10^{-K_1 t} = 0.435$
$K_2 - K_1 = 0.076$ diff. $= 0.163$ $a = \frac{0.335}{0.076} = 4.41$

$D_a = 4.41 \times 0.163 +$
$+ 4.00 \times 0.435 =$
$= 0.718 + 1.74 =$
$= 2.46$

Plot above corresponding values of (K_2) and (D_a) and take off value of (K_2) corresponding to $D_a = 1.32$. Thus, $K_2 = 0.38$.

To confirm $K_2 = 0.38$.

Try $K_2 = 0.380$ $10^{-K_2 t} = 0.598$

$K_1 = 0.124$ $10^{-K_1 t} = 0.206$
$K_2 - K_1 = 0.256$ diff. $= 0.392$ $a = \frac{0.335}{0.256} = 1.31$

$D_a = 1.31 \times 392 +$
$+ 4.00 \times 0.206 =$
$= 0.51 + 0.81 =$
$= 1.31$

This value checks that of (K_2) from the curve.

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